Ex Post Implementation *

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Abstract

In this paper we investigate the *full* implementation problem under conditions of incomplete information. The solution concept we use is *ex post equilibrium*. If a social choice set X is fully implementable in ex post equilibrium, it will call that X is *ex post implementable*. We provide the necessary and almost sufficient condition for ex post implementation. Our approach to the implementation is follows that used by Mookherjee and Reichelstein (1990, hereafter MR), who investigated the Bayesian implementation via *augmented mechanism*. We show that *the Ex Post Selective Elimination* (EPSE) condition, which is analogous to the Selective Elimination condition defined in MR, and ex post incentive compatibility (EPIC)

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are necessary conditions for which a social choice set X is ex post implementable. Moreover, a social choice set X is ex post implementable if both the EPSE and EPIC conditions are satisfied in an economic environment.

1 Introduction

The purpose of this paper is to investigate the full implementation of a *social choice set* in ex post equilibrium under conditions of incomplete information and general interdependent values. The theory of implementation has been investigated under several environments, such as implementation in Nash equilibrium in a complete information environment (e.g., Maskin(1999)), and implementation in Bayesian equilibrium in an incomplete information (e.g., Jackson(1991), Mookherjee and Reichelstein (1990)). In this paper, we focus on an incomplete information environment. The typical solution concept of implementation in an incomplete information environment is a Bayesian equilibrium. We should point out, however, that the theory of Bayesian implementation, or more generally, the implementation problem under incomplete information, has assumed explicitly (or implicitly) that the planner knows the belief distribution of agents and she can use this information to design the mechanism which implements a social choice function (or social choice set). This assumption may be sometimes unrealistic.

In this paper we consider the implementation problem without the assumption that a planner has full knowledge about prior distribution of types of agents on their belief systems. The solution concept we use in the paper is ex post equilibrium, which can be seen as a Bayesian equilibrium, subject to a "no regret" condition — a formal definition of ex post equilibrium will be given in Section 2. This solution concept is stronger than that proposed by a Bayesian equilibrium, but the planner who knows nothing about the belief distribution of types of agents *may* implement a social choice function if the planner uses ex post equilibrium as the solution concept.

The main results of this paper provides the necessary and almost sufficient condition for implementation of social choice set *in ex post equilibrium*, which we will refer to as *ex post implementation*. The related literature has shown that it is necessary to satisfy an *incentive compatibility* condition in order for a social choice function to be implemented. This result is well known as the *revelation principle*. It is also well known, however, that the revelation principle itself cannot guarantee the full implementability of a social choice function. Mookherjee and Reichelstein (1990) introduced the idea of *augmented revelation* and showed that incentive compatibility[1] and *selective elimination* (*SE*) conditions are necessarily held when a social choice *correspondence* is *Bayesian implementable*. [2] They also showed that the converse is true when one consider the implementation of a social choice *function* and the environment is economic.

Our approach to the ex post implementation follows Mookherjee and Reichelstein and the results of this paper are analogous to their results. We define the *ex post selective elimination* condition, which is our own version of the *se-*

[1]Strictly speaking, It should be said the Bayesian incentive compatibility.

[2]They referred that their result of necessary condition can be expanded to the case of implementation of a social choice set.

lective elimination condition, and show that the ex post incentive compatibility and ex post selective elimination condition are necessary for ex post implementation of a social choice set. Conversely, both conditions are sufficient to implement a social choice set if the environment allows agents to transfer private goods between themselves.

The paper is organized as follows. Section 2 introduces the model and definitions. Section 3 provides the necessary condition for ex post implementation and discusses the relationship between the ex post selective elimination condition and the (original) selective elimination condition by example. Section 4 provides the sufficient condition for ex post implementation.

2 Model

We first describe the general environment $< A, N, \Theta >$ that will be taken into account. A is the set of alternatives or outcomes, $N = \{1, \cdots, n\}$ is the finite set of agents, and $\Theta = \Theta_1 \times \cdots \times \Theta_n$ is the set of possible states of the world, with a typical state denote as $\theta \in \Theta$. We describe each agent as $i \in N$ and assume that $n \geq 2$. Θ_i is the set of payoff relevant types for agent $i \in N$, and describe the typical type of i by $\theta_i \in \Theta_i$. We assume that each Θ_i is a finite set. Preference of each agent is represented by the utility function $u_i : A \times \Theta \to \mathbb{R}$. In addition we assume that u_i satisfies an expected utility property (i.e., u_i is von Neumann-Morgenstern utility function).

The function $x : \Theta \to A$ is a social choice function. The *mechanism* is denoted by $\Gamma =< M, g >$ and consists of the Cartesian product of each agent's message space: $M = M_1 \times \cdots \times M_n$ and an outcome function $g : M \to A$. We denote a typical message by $m = (m_i, m_{-i}) \in M$. Given the mechanism $\Gamma =< M, g >$, we define a game $G = (\Gamma, (u_i)_{i=1}^n)$ with the environment.[3] Given an arbitrary game G, we assume that the planner and agents have common knowledge about the structure of G. Let $\alpha_i : \Theta_i \to M_i$ denote a *pure strategy* for $i \in N$ in G. We describe a typical strategy profile by $\alpha = (\alpha_i, \alpha_{-i}) = (\alpha_1, \cdots, \alpha_n)$.

Definition 1. Given a game $G = (\Gamma, (u_i)_{i=1}^n)$, a pure strategy profile α is an *ex post equilibrium* in G if for all $i \in N$, for all $\theta \in \Theta$ and for all $m_i \in M_i$,

$$u_i(g(\alpha(\theta)), \theta) \geq u_i(g(m_i, \alpha_{-i}(\theta_{-i})), \theta). \qquad (1)$$

For notational convenience, we will often write this as $g(\alpha(\theta)) = g \circ \alpha$. A *social choice set (SCS)* $X = \{x_\lambda\}_{\lambda \in \Lambda}$ is a collection of social choice functions with an arbitrary index Λ. We denote $x \in X$ if and only if there exists $\lambda \in \Lambda$ such that $x_\lambda(\theta) = x(\theta)$ for all $\theta \in \Theta$.

Definition 2. A mechanism $\Gamma =< M, g >$ *implements a SCS X in ex post equilibrium* in $G = (\Gamma, (u_i)_{i=1}^n)$ if the following both statements are satisfied:

[3]Note that the term "game" is different from "game form", which is often used as the same meaning of mechanism. The reason why we use the term "game" is that we will compare two games: one of which is a game $G = (\Gamma, (u_i)_{i=1}^n)$, the other is the game on which we assume common prior about type distributions. See the Section 3.1.

1. for every *pure* ex post equilibrium α in G, it is true that $g \circ \alpha = x_\lambda$ for some $\lambda \in \Lambda$.

2. for any $x \in X$ there exists an *pure* ex post equilibrium α in G such that $g \circ \alpha = x$.

In this case, we will say that X is *ex post implementable*. Throughout this paper, we only consider the *pure* strategy equilibrium.[4]

Mookherjee and Reichelstein (1990) provided the mechanism whose message space is defined by union of the payoff relevant type space and an arbitrary set. They called such a mechanism the *augmented revelation mechanism*. Strictly speaking, an augmented revelation mechanism is said to be a mechanism $\Gamma =< M, g >$ such that for all $i \in N$:

$$M_i = \Theta_i \cup T_i, \tag{2}$$

where T_i is an arbitrary set and it allows to be empty for some $i \in N$. Mookherjee and Reichelstein introduced the augmented revelation principle, which states that if a social choice set is Bayesian implementable via some arbitrary mechanism, then it is also Bayesian implementable via augmented revelation mechanism. This statement is applicable to our solution concept.

[4] Allowing mixed strategy ex post equilibria may be against the implementability of social choice set. See Section 4.3.

Proposition 1. *If SCS X is ex post implementable, then X can be ex post implemented by an augmented revelation mechanism, in which truthful reporting is an ex post equilibrium.*

Proof. See Mookherjee and Reichelstein (1990). ■

3 Necessary Condition for Ex Post Implementation

In this section, we will provide the necessary condition for ex post implementation. Our condition is analogous to Mookherjee and Reichelstein.

We denote by $\Gamma_x^d =< \theta, x >$ a direct revelation mechanism relative to x. Γ_x^d is said to be *ex post incentive compatible* if for all $i \in N$, for all $\theta \in \Theta$ and for all $\theta_i' \in \Theta_i$,

$$u_i(x(\theta), \theta) \geq u_i(x(\theta_i', \theta_{-i}), \theta). \tag{3}$$

Let $G_x^d = (\Gamma_x^d, (u_i)_{i=1}^n)$ be a game with a mechanism Γ_x^d.

Definition 3. An ex post equilibrium $\alpha = (\alpha_1, \cdots, \alpha_n)$ in G_x^d can be *ex post selectively eliminated*, if there exists $i^* \in N$ and another social choice function $y : \Theta_{-i^*} \to A$ such that:

$$\exists \theta \in \Theta, \;\; u_{i^*}(y(\alpha_{-i^*}(\theta_{-i^*})), \theta) > u_{i^*}(x(\alpha(\theta)), \theta) \tag{4}$$

and

$$\forall \theta \in \Theta, \;\; u_{i^*}(x(\theta), \theta) \geq u_{i^*}(y(\theta_{-i^*}), \theta). \tag{5}$$

We call agent i^* the *whistle-blower* at profile α in G_x^d. We say that a direct revelation mechanism Γ_x^d satisfies *the ex post selective elimination* (hereafter *EPSE*) *condition relative to* X if $x \in X$ and if any ex post equilibrium α in G_x^d which satisfies $x \circ \alpha \notin X$ can be ex post selectively eliminated.

Proposition 2. *If SCS X is ex post implementable, then, for any $x \in X$, there exists an ex post incentive compatible direct revelation mechanism Γ_x^d which satisfies the EPSE condition relative to X.*

Proof. Since X is ex post implementable, and by the Proposition 1, we only focus on the augmented revelation mechanism $\Gamma =< M_1 \times \cdots \times M_n, g >$ which ex post implements X. We will denote the game with this augmented revelation mechanism by $\hat{G} = (\Gamma, (u_i)_{i=1}^n)$. Let $g|_\Theta$ be an outcome function of $\hat{G}$ whose domain is restricted to Θ. Take an arbitrary $x \in X$ and consider the direct revelation mechanism $\Gamma_x^d =< \Theta, x >$ with $x = g|_\Theta$. Truthful type reporting profile is an ex post equilibrium in $\hat{G}$, and in $G_x^d = (\Gamma_x^d, (u_i)_{i=1}^n)$ because of the Proposition 1. Thus Γ_x^d is ex post incentive compatible.

Suppose that there exists an ex post equilibrium α in G_x^d such that $x \circ \alpha \notin X$. Note that $g(\alpha(\theta)) = x(\alpha(\theta))$. Since Γ ex post implements X, we must have

$$\neg\left[\ \forall i \in N, \forall \theta \in \Theta, \forall m_i \in M_i,\ \ u_i\big(g(\alpha(\theta)), \theta\big) \geq u_i\big(g(m_i, \alpha_{-i}(\theta_{-i})), \theta\big)\ \right]. \quad (6)$$

That is,

$$\exists i \in N, \exists \theta \in \Theta, \exists \hat{m}_i \in M_i,\ \ u_i\big(g(\alpha(\theta)), \theta\big) < u_i\big(g(\hat{m}_i, \alpha_{-i}(\theta_{-i})), \theta\big). \quad (7)$$

In addition, by the Proposition 1, we must have

$$\forall \theta \in \Theta, \;\; u_i(g(\theta), \theta) \geq u_i(g(\hat{m}_i, \theta_{-i}), \theta). \tag{8}$$

Define $y(\theta_{-i}) := g(\hat{m}_i, \theta_{-i})$, so we have a function $y : \Theta_{-i} \to A$. Moreover, we must have $\hat{m}_i \notin \Theta_i$. Suppose that $\hat{m}_i \in \Theta_i$. Let $\hat{\alpha}_i$ be a strategy such that $\hat{\alpha}_i(\theta_i) = \hat{m}_i$ for some $\theta_i \in \Theta_i$. Since $g|_\Theta = x$, we must have $g(\hat{m}_i, \alpha_{-i}(\theta_{-i})) = x(\hat{\alpha}_i(\theta_i), \alpha_{-i}(\theta_{-i}))$ and this satisfies (7). This is contrary to the assumption that α is an ex post equilibrium in G_x^d. Then, this proof is complete. ■

Note that the social choice function y which is defined above must satisfy $y \notin X$ if X is a singleton. Let $x(\Theta_i|\theta_{-i})$ denote possible values of x subject to θ_{-i} is fixed arbitrarily. Suppose that X is a singleton, i.e., $X = \{x\}$ and x is ex post implementable. Let α be an ex post equilibrium in G_x^d with $x \circ \alpha \neq x$. Then we claim that the social choice function y defined above must satisfy $y(\theta'_{-i}) \notin x(\Theta_i|\theta'_{-i})$ for some θ'_{-i}. Suppose that $y(\theta'_{-i}) \in x(\Theta_i|\theta'_{-i})$ for all θ'_{-i}. Since α is an ex post equilibrium in Γ_x^d, we must have

$$u_i(x(\alpha(\theta)), \theta) = \max_{\hat{\theta}_i \in \Theta_i} u_i(x(\hat{\theta}_i, \alpha_{-i}(\theta_{-i})), \theta). \tag{9}$$

Thus, holding $y(\alpha_{-i}(\theta_{-i})) \in x(\Theta_i|\alpha_{-i}(\theta_{-i}))$ for all θ_{-i} is contrary to the equation (7). This implies that $y \notin X$ if X is singleton. However, this fact is not necessarily held if X is not a singleton. We consider the following example:

Example 1. *A social choice function $y : \Theta_{-i^*} \to A$ can be $y \in X$ if X is not a singleton, where i^* is a whistle-blower in G_x^d with $x \in X$.*

Consider the following environment: $N = \{1, 2\}$, $A = \{a, b, c, d, e\}$ and $\Theta = \Theta_1 \times \Theta_2$ with $\Theta_i = \{\theta_i^1, \theta_i^2\}$ for each $i = 1, 2$. Suppose that the social choice set is $X = \{x, y\}$. Each element of X is a function and these values are described in Table 1.[5]

— Table 1 —

x	θ_2^1	θ_2^2
θ_1^1	a	b
θ_1^2	c	d

y	θ_2^1	θ_2^2
θ_1^1	e	e
θ_1^2	e	e

We assume that each agent has the same preference. In each state, their payoffs are as follows:

— Table 2 —

a	θ_2^1	θ_2^2
θ_1^1	3, 3	0, 0
θ_1^2	0, 0	1, 1

b	θ_2^1	θ_2^2
θ_1^1	0, 0	3, 3
θ_1^2	1, 1	0, 0

c	θ_2^1	θ_2^2
θ_1^1	0, 0	1, 1
θ_1^2	3, 3	0, 0

d	θ_2^1	θ_2^2
θ_1^1	1, 1	0, 0
θ_1^2	0, 0	3, 3

[5] In this table, for example, one can see that the preference of each agent at state (θ_1^1, θ_2^1) is orderd as $a \succ e \succ d \succ b \sim c$.

e	θ_2^1	θ_2^2
θ_1^1	$2,2$	$2,2$
θ_1^2	$2,2$	$2,2$

Note that the direct mechanism $\Gamma_x^d = (\Theta, x)$ has a suboptimal ex post equilibrium $\check{\alpha}$ in the game $G_x^d = (\Gamma_x^d, (u_i)_{i=1,2}) : \check{\alpha}_1(\theta_1^1) = \theta_1^2$, $\check{\alpha}_1(\theta_1^2) = \theta_1^1$, $\check{\alpha}_2(\theta_2^1) = \theta_2^2$, $\check{\alpha}_2(\theta_2^2) = \theta_2^1$. Each agent's strategy is to mimic the other type of him. Moreover Γ_x^d satisfies the EPSE condition relative to X in this environment: $u_i(x(\check{\alpha}(\theta)), \theta) < u_i(y(\check{\alpha}_j(\theta_j)), \theta)$ and $u_i(x(\theta), \theta) \geq u_i(y(\theta_j), \theta)$ where $i \neq j$ and $y(\theta_j) = e$ for all $\theta_j \in \Theta_j$.

In this situation, we set the mechanism $\Gamma =< M, g >$ whose message spaces of agents are $M_1 = \Theta_i \cup \{r_1\}$ and $M_2 = \Theta_2$, and values of outcome function g are described as follows;

g	θ_2^1	θ_2^2
θ_1^1	a	b
θ_1^2	c	d
r_1	e	e

If we define the game $G = (\Gamma, (u_i)_{i=1,2})$ with this mechanism Γ, then Γ implements X in ex post equilibria in G. Note that the social choice function whose outcomes are prefered to the suboptimal equilibrium outcome in G_x^d is $y \in X$.

3.1 Relationship between the EPSE and SE condition

In this subsection, we discuss the relationship between the EPSE and SE (i.e., Selective Elimination) condition by example. Mookherjee and Reichelstein showed that the SE condition is necessary for Bayesian implementation. Consider a direct mechanism Γ_x^d relative to X and X is Bayesian implementable. Then, Γ_x^d must satisfy the SE condition. Does Γ_x^d also satisfy the EPSE condition ? This is a trivial question and the answer is "Yes", as we will consider later in this section. Conversely, if Γ_x^d satisfies the EPSE condition, does Γ_x^d also satisfy the SE condition ? This is the other question in this section and we show by example that the answer is "No".

Let $BG = (\Gamma, (u_i)_{i=1}^n, p)$ denote a Bayesian game with some mechanism Γ and common prior $p : \Theta \to [0,1]$.

Definition 4. A Bayesian equilibrium $\alpha = (\alpha_1, \cdots, \alpha_n)$ in a game $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ can be *selectively eliminated*, if there exists $i \in N$ and $y : \Theta_{-i} \to A$ such that:

for some $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i}|\theta_i) \left[u_i(y(\alpha_{-i}(\theta_{-i})), \theta) - u_i(x(\alpha(\theta)), \theta) \right] > 0 \tag{10}$$

and for all $\theta_i \in \Theta_i$,

$$\sum_{\theta_{-i} \in \Theta_{-i}} p_i(\theta_{-i}|\theta_i) \left[u_i(x(\theta), \theta) - u_i(y(\theta_{-i}), \theta) \right] \geq 0. \tag{11}$$

We will say that a direct revelation mechanism $\Gamma_x^d = <\Theta, x>$ satisfies *the selective elimination (SE) condition relative to* X if $x \in X$ and if any Bayesian equilibrium α in $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ such that $x \circ \alpha \notin X$ can be selectively eliminated.

We can check easily that the definition of SE condition implies that if the direct revelation mechanism Γ_x^d satisfies the SE condition relative to X, then Γ_x^d also satisfies the EPSE condition when we consider the game $G_x^d = (\Gamma_x^d, (u_i)_{i=1}^n)$.[6]

That a Γ_x^d satisfies the EPSE condition, however, does not imply that the Γ_x^d also satisfies the SE condition in a game $BG = (\Gamma_x^d, (u_i)_{i=1}^n, p)$. We show this fact by the following example.

Example 2. *The EPSE condition relative to X can be satisfied even if the SE condition is not satisfied.*

Consider the environment $N = \{1, 2\}$, $A = \{a^{11}, a^{12}, a^{21}, a^{22}, b, c\}$ and $\Theta = \Theta_1 \times \Theta_2$ with $\Theta_i = \{\theta_i^1, \theta_i^2\}$ for each $i = 1, 2$. Assume that we want to implement the following social choice function x:

— Table 3 —

[6]It is possible that there does not exist any ex post equilibria in $G = (\Gamma_x^d, (u_i)_{i=1}^n)$ even though there exists many Bayesian equilibria in $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^n, p)$ which is defined with some common prior p. In such a case, we should interpret that the EPSE condition for Γ_x^d is vacuously satisfied.

x	θ_2^1	θ_2^2
θ_1^1	a^{11}	a^{12}
θ_1^2	a^{21}	a^{22}

In each state, their ex post payoffs (von Neumann-Morgenstern utilities) are as follows:

— Table 4 —

a^{11}	θ_2^1	θ_2^2
θ_1^1	B, B	$0, -L$
θ_1^2	$-L, 0$	$0, -L$

a^{12}	θ_2^1	θ_2^2
θ_1^1	$0, -L$	$0, -L$
θ_1^2	$B, -L$	$-L, 0$

a^{21}	θ_2^1	θ_2^2
θ_1^1	$-L, 0$	$0, -L$
θ_1^2	B, B	$0, -L$

a^{22}	θ_2^1	θ_2^2
θ_1^1	$B, -L$	$-L, 0$
θ_1^2	$0, -L$	$0, -L$

b	θ_2^1	θ_2^2
θ_1^1	$-M, 0$	$D, 0$
θ_1^2	$0, 0$	$0, 0$

We assume that $D > B > M > L > 0$ and $B > D - M$. The direct mechanism Γ_x^d induces a truthful ex post equilibrium in G_x^d and also induces a suboptimal ex post equilibrium. The suboptimal ex post equilibrium profile is

$$\Big(\alpha_1(\theta_1^1), \alpha_1(\theta_1^2), \alpha_2(\theta_2^1), \alpha_2(\theta_2^2)\Big) = \Big(\theta_1^1, \theta_1^2, \theta_2^1, \theta_2^1\Big). \tag{12}$$

In addition, the EPSE condition relative to x is also satisfied in this case. Define the social choice function as follows: $y(\theta_2^1) = b$ and $y(\theta_2^2) = a^{22}$, we have that $u_1(x(\alpha(\theta_1^1, \theta_2^2)), (\theta_1^1, \theta_2^2)) < u_1(y(\alpha_2(\theta_2^2)), (\theta_1^1, \theta_2^2))$ and $u_1(x(\theta), \theta) \geq u_1(y(\theta_2), \theta)$ for all $\theta \in \Theta$.

On the other hand, consider the Bayesian environment, i.e., there is the common prior distribution relating to types of agents and this fact is common knowledge between the planers and agents. We assume that the distribution of type of agents is $p(\theta_i) = \frac{1}{2}$ for all $\theta_i \in \Theta_i$ $(i = 1, 2)$. In this environment, Γ_x^d does not satisfy the SE condition when we consider the game $BG_x^d = (\Gamma_x^d, (u_i)_{i=1}^2, p)$. Suppose that Γ_x^d satisfy the SE condition. Let agent 1 be a whistle-blower, then his payoff satisfies the following equation:

$$\frac{1}{2}(C^{11} - B) + \frac{1}{2}C^{12} > 0 \tag{13}$$

or

$$\frac{1}{2}(C^{21} - B) + \frac{1}{2}C^{22} > 0 \tag{14}$$

where C^{kl} corresponds to the payoff for the agent 1 from the social choice function $y : \Theta_2 \to A$ at the state (θ_1^k, θ_2^l) $(k, l = 1, 2)$. However, one can easily check that it is impossible for any feasible social choice functions y to exist which give such payoffs for agent 1 at each state. By the same reasoning, we can say that agent 2 cannot be a whistle-blower.

4 Sufficient Condition for Ex Post Implementation

In this section, we will consider the sufficient condition for ex post implementation of a social choice set X. We will show that the EPSE and EPIC condition play a main role in our sufficiency theorem if the environment is economic, i.e., an environment in which we allows agents to transfer private goods among themselves.

4.1 The Economic Environment

We will consider the environment in which each agent and planner can trade private goods, e.g., money. We define the following economic environment: the set of socially feasible alternatives is $A = \bar{A} \times \mathbb{R}^n$, and a social choice set $X = \{x_\lambda\}_{\lambda \in \Lambda}$ is a collection of the social choice function $x_\lambda : \Theta \to A$ with an arbitrary index $\lambda \in \Lambda$. Let $x \in X$ denote an arbitrary social choice function which belongs to X. Any social choice function x is denoted by $x = (\bar{x}, (t_i)_{i=1}^n)$, where $\bar{x} : \Theta \to \bar{A}$ is said to be a *public decision rule* and $t_i : \Theta \to \mathbb{R}$ to be a *transfer function* for i. We denote $t = (t_i)_{i=1}^n$. In addition we assume that each agent has a quasi-linear utility function: $u_i((a,t),\theta) = v_i(a,\theta) + t_i$ for every $a \in A$, $t \in \mathbb{R}^n$ and $\theta \in \Theta$, and also assume that $v_i(a,\theta)$ is a bounded function for every $i \in N$.

In our economic environment, the outcome function of a mechanism $\Gamma =< M, g >$ can be denoted by $g(m) = (g_{\bar{x}}(m), (g_{t_i}(m))_{i=1}^n)$. We also use the notation such that $g(m) = ((g_i(m))_{i=1}^n)$ where $g_i(m) = (g_{\bar{x}}(m), g_{t_i}(m))$ for every

$m \in M$ and $i \in N$.

A direct mechanism Γ_x^d satisfies the *ex post incentive compatible* (*EPIC*) condition relative to X if $x \in X$ and it is ex post incentive compatible. The definition of EPSE condition is the same one in Definition 3; however, note that a social choice function y in Definition 3 should be denoted by a collection $y = (\bar{y}, (\xi_j)_{j=1}^n)$ and each element is defined respectively as follows: $\bar{y} : \Theta_{-i} \to A$, $\xi_j : \Theta_{-i} \to \mathbb{R}$ in the economic environment we defined.

4.2 Result

Proposition 3. *In the economic environment, a social choice set X is ex post implementable if Γ_x^d satisfies both the EPSE and EPIC conditions relative to X.*

Proof. Our proof is constructive. Let N^λ be a set of whistle-blowers in $G_{x_\lambda}^d$, that is,

$$N^\lambda = \{i \in N : i \text{ is a whistle-blower at some ex post equilibrium profile } \alpha \text{ in } G_{x_\lambda}^d.\} \tag{15}$$

We denote $N^* = \cup_{\lambda \in \Lambda} N^\lambda$. By assumption, for any $x_\lambda \in X$ and for every ex post equilibrium in $G_{x_\lambda}^d$, there exists a whistle-blower $i \in N^\lambda$ and another choice function which satisfies the equation (4) and (5). We will denote by $y_i^{(\alpha,\lambda)} : \Theta_{-i} \to A$ a social choice function if and only if $y_i^{(\alpha,\lambda)}$ satisfies (4) and (5), α is an ex post equilibrium profile in $G_{x_\lambda}^d$ such that $x_\lambda \circ \alpha \notin X$ and i is a whistle-blower at an ex post equilibrium profile α in $G_{x_\lambda}^d$. It is possible that agent i is also a whistle-blower at another ex post equilibrium profile in $G_{x_\lambda}^d$

or at an ex post equilibrium profile in $G^d_{x_{\lambda'}}$ $(\lambda \neq \lambda')$. Fix $\lambda \in \Lambda$ arbitrarily. Without loss of generality, we can say that $G^d_{x_\lambda}$ has the amount of K_λ ex post equilibria which satisfy $x_\lambda \circ \alpha \notin X$. We will construct message space for agents inductively as follows. For every $i \in N$ we set

$$M_i^{(1,\lambda)} = \Big(\Theta_i \times \Lambda\Big) \cup \{1, \cdots, n\}. \tag{16}$$

Let α^k be an k-th ex post equilibrium in $G^d_{x_\lambda}$ such that $x_\lambda \circ \alpha^k \notin X$, then there exists a whistle-blower $i \in N^\lambda$ at α^k in $G^d_{x_\lambda}$. If there are many whistle-blowers at α^k, then we choose only one arbitrarily from such agents and set

$$M_i^{(k+1,\lambda)} = M_i^{(k,\lambda)} \cup F^{(k,\lambda)} \tag{17}$$

for the chosen whistle-blower where $F^{(k,\lambda)}$ is an arbitrary message and satisfy $M_i^{(k,\lambda)} \cap F^{(k,\lambda)} = \emptyset$. We do this inductive augmentation of message space from $k = 1$ to $k = K_\lambda$. Note that the augmentation of message space from k to $k+1$ implies that there exists only one agent $i \in N^\lambda$ whose message space comes to $|M_i^{(k+1,\lambda)}| = |M_i^{(k,\lambda)}| + 1$ where $|M|$ means the cardinality of a set M and the message spaces of the others are $M_j^{(k+1,\lambda)} = M_j^{(k,\lambda)}$ $(j \neq i)$. The message spaces especially to the agents $j \in N \backslash N^\lambda$ are $M_j^{(k,\lambda)} = M_j^{(1,\lambda)}$ for all k. Denote $M_i^\lambda := M_i^{(K_\lambda+1,\lambda)}$, $M_i := \cup_{\lambda\in\Lambda} M_i^\lambda$ for all $i \in N$, $\mathcal{F}^\lambda = \{F^1, \cdots, F^{K_\lambda}\}$, and $\mathcal{F} = \cup_{\lambda\in\Lambda}\mathcal{F}^\lambda$. For notational ease, if $m \in \Theta \times \Lambda$, then we will write $m_i = \alpha_i(\theta_i) = (\alpha_i^1(\theta_i), \alpha_i^2(\theta_i)) = (m_i^1, m_i^2)$ for every $i \in N$. Let $a = (a_i, (b)^{n-1})$ denote an element of $\mathbb{R}^n$ if $a_j = b$ for all $j \neq i$.

Let the outcome function be as follows;

Rule 1. If $m \in \Theta \times \Lambda$ and $m_i^1 \in \Theta_i$, $m_i^2 = \lambda$ for all i, then

$$g(m) = g(m_1^1, \cdots, m_n^1) = x_\lambda(\hat{\theta}). \tag{18}$$

Rule 2. If $m_i = F^{(k,\lambda)}$ for some k, λ and $m_{-i} \in \Theta_{-i} \times \Lambda$ such that $m_{-i}^1 = \hat{\theta}_{-i} \in \Theta_{-i}$, $m_j^2 = \lambda$ for all $j \in N \backslash \{i\}$, then

$$g(m) = y_i^{(\alpha^k,\lambda)}(\hat{\theta}_{-i}). \tag{19}$$

Rule 3. If $m_i \in \{1, \cdots, n\}$ and $m_{-i} \in \Theta_{-i} \times \Lambda$, then

$$g(m) = (g_i(m), g_{-i}(m)) = \Big((z, -\frac{\delta}{2}), (z, \frac{\delta}{2(n-1)})^{n-1} \Big), \tag{20}$$

where $z \in A$ is some arbitrary alternative and we take $\delta > 0$ such that for all $x \in X$, for all ex post equilibrium α with $x \neq x \circ \alpha$, and for all $(\theta, \lambda, i, j, \theta') \in \Theta \times \Lambda \times N \times N \times \Theta$,

$$v_j(\bar{y}_i^{(\alpha,\lambda)}(\theta'_{-i}), \theta) + \xi_j(\theta'_{-i}) - v_j(z, \theta) < \delta. \tag{21}$$

Rule 4. If there are at least two agents i, j $(i \neq j)$ such that $m_i, m_j \notin \Theta \times \Lambda$, then

$$g(m) = (g_h(m), g_{-h}(m)) = \Big((z, \delta), (z, \frac{-\delta}{n-1})^{n-1} \Big), \tag{22}$$

where z and δ in (22) are the same ones in (20). In the equation (22), h is the agent who matches up $h = \text{mod}_n \sum_{l=1}^n r_l(m_l)$, where $r_l : M_l \to \{0, 1, \cdots, n\}$ with its value

$$r_l(m_l) = \begin{cases} k & \text{if } m_l = k \in \{1, \cdots, n\} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

We call such an agent the *MG winner* (which means modulo game winner). We make the agent n be the MG winner if $h = 0$. We can check easily that a MG winner is always determined uniquely for any profile $m \in M$.[7]

Rule 5. If $m \in \Theta \times \Lambda$ and $m_i^2 \neq m_j^2$ for some i, j ($i \neq j$), then

$$g_i(m) = (z, -\delta) \tag{24}$$

for all $i \in N$.

Consider the game $G = (\Gamma, (u_i)_{i=1}^n)$ with $\Gamma =< M, g >$ where $M = M_1 \times \cdots \times M_n$ and g follow the above settings. Our proof is complete if we show the following four claims;

(1) *There are no pure ex post equilibrium profile α such that $g \circ \alpha(\theta) \neq x_\lambda(\theta)$ with $\alpha(\theta) \in \Theta \times \Lambda$ for all $\theta \in \Theta$ and $m_i^2 = \lambda$ for all $i \in N$.*

It is trivial from the Rule 1, Rule 2, EPSE and EPIC condition relative to x. $\|$

(2) *There are no pure ex post equilibrium profile α such that $\alpha(\theta) \in \Theta \times \Lambda$ for all $\theta \in \Theta$ and $\alpha_i^2(\tilde{\theta}_i) \neq \alpha_j^2(\tilde{\theta}_j)$ for some $i, j \in N$ ($i \neq j$) at some $\tilde{\theta}_i$, $\tilde{\theta}_j$.*

Consider such a message profile $m = \alpha(\tilde{\theta}_i, \tilde{\theta}_j, \theta_{-i,j})$. Then Rule 5 applies, so the outcome for each agent is $g_l(m) = (z, -\delta)$. Any agent $l \in N$ wants to deviate unilaterally from this message profile by changing his message into

[7] Let *i,j* be MG winners at message profile m. Denote $r_l = r_l(m_l)$. Then there exists $K \in \mathbb{N}$ such that $\sum_{l=1}^n r_l = nK + i = nK + j$. Thus $i = j$.

$m_l \in \{1, \cdots, n\}$ at the state $(\tilde{\theta}_i, \tilde{\theta}_j, \theta_{-i,j})$ because he can get higher payoff by using Rule 3. Thus this strategy profile cannot be an ex post equilibrium. $\|$

(3) *There are no pure ex post equilibrium profile α such that some agent $i \in N$ reports non-type message and a message profile of the others is in $\Theta_{-i} \times \Lambda$.*

Let α be a strategy profile such that an agent i reports a non-type message. Denote $m_{-i} = \alpha_{-i}(\theta_{-i})$. Suppose that $\alpha_i(\tilde{\theta}_i) \in \mathcal{F}$ and $m_{-i} \in \Theta_{-i} \times \Lambda$. Then Rule 2 applies, so there is an agent $j \neq i$ who wants to deviate unilaterally from this message profile at the state $\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$ by changing his message into $m_j \in \{1, \cdots, n\}$ because he can get higher payoff by Rule 4. Next suppose that $\alpha_i(\tilde{\theta}_i) \in \{1, \cdots, n\}$ and $m_{-i} \in \Theta_{-i} \times \Lambda$. Then Rule 3 applies, so there is an agent $j \neq i$ who wants to be the MG winner, because he has a chance to get a higher payoff by changing his message appropriately. $\|$

(4) *There are no pure ex post equilibrium profile such that at least two agents report a non-type message.*

Let α be a strategy profile such that some agents i, j ($i \neq j$) report a non-type message. Denote $m_{-i,j} = \alpha_{-i,j}(\theta_{-i,j})$. Suppose that $\alpha_i(\tilde{\theta}_i) \in \mathcal{F}$ and $\alpha_j(\tilde{\theta}_j) \in \mathcal{F}$. Whatever $m_{-i,j}$ be, there exists the MG winner by Rule 4. Any agent except for the MG winner wants to deviate unilaterally from this message profile at the state $\tilde{\theta} = (\tilde{\theta}_i, \tilde{\theta}_j, \theta_{-i,j})$. The same reasoning applies to the case of

$\left(\alpha_i(\tilde{\theta}_i), \alpha_j(\tilde{\theta}_j)\right) \in \mathcal{F} \times \{1, \cdots, n\}$ and $\left(\alpha_i(\tilde{\theta}_i), \alpha_j(\tilde{\theta}_j)\right) \in \{1, \cdots, n\} \times \{1, \cdots, n\}$.■

4.3 Note on Proposition 3

Our result of sufficient condition for ex post implementation allow for the case of $n = 2$. It studies on implementation have often assumed $n \geq 3$. Bergemann and Morris (2003) study ex post implementation of social choice *function* and provide similar results as ours. However, their result for sufficient condition for ex post implementation is derived with the assumption $n \geq 3$.

Note that we only consider pure strategy ex post equilibria. If we allow for using mixed strategy, this mechanism brings about some mixed strategy ex post equilibria in the game. We denote by $\beta_i : \Theta_i \to \triangle_{M_i}$ a mixed strategy for an agent i where $\triangle_{M_i}$ is the set of probability distribution on M_i. We can check it with a simple example: In the case of $n = 2$, the following strategy profile forms a mixed ex post equilibrium: for all $i = 1, 2$, for all $\theta \in \Theta$, $\beta_i(\theta_i)$ is the strategy such that $\beta_i(\theta_i) = k$ $(k = 1, 2)$ with probability $\frac{1}{2}$. Each agent can get a payoff at most $v_i(z, \theta)$ for all $\theta \in \Theta$.

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