Role of Linking Mechanisms in Multitask Agency with Hidden Information[1]

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Abstract

We investigate the adverse selection problem where a principal delegates multiple tasks to individuals. The individuals form a group as a single agent and share their private signals in order to maximize their average payoff. We characterize the virtually implementable social choice functions by using the linking mechanism proposed by Jackson and Sonnenschein (2005) that restricts the message spaces. The principal does not require any incentive wage schemes and can therefore avoid any information rent and welfare loss due to risk aversion. We show the resemblance between the functioning of this message space restriction and that of incentive wage schemes.

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1. Introduction

This paper investigates the adverse selection problem in which a principal (an employer) hires many individuals (employees) and delegates different tasks to each of them; these tasks are assumed to be independent of each other and homogeneous. The hired individuals observe the private signals relevant to their respective tasks; however, the principal cannot observe these signals. Therefore, the principal will attempt to incentivize these individuals to announce their true private signals by designing a well-behaved mechanism or a contract.

The standard approach in the informational economics literature is that the principal regards each individual as a single selfish agent and designs separate wage schemes for each person. These schemes base wage payments on the individual's announcements. However, this approach has the following two drawbacks. First, if the lower bound of wage payments (such as nonnegativity) exists, each individual can earn a positive information rent and the principal fails to extract the full surplus in a non-negligible manner. Second, if individuals are risk averse, a welfare loss is inevitable because the inconstant wage schemes shift the burden of risk sharing onto these risk-averse individuals.

Based on these observations, this paper presents an alternative approach to solve the adverse selection problem and suggests a means to overcome the above mentioned drawbacks. In other words, the principal compels the individuals to establish a well-coordinated working group in which they share their private signals and agree to maximize the average of their payoffs. The principal regards this group as a single agent with whom she makes a contract. There exists a serious conflict of interest between the principal and this group. However, by contracting with the group as a whole, the principal is not required to design an inconstant wage scheme and succeeds in extracting the full surplus without suffering any non-negligible welfare distortion. More precisely, this paper will show that when the size of the group (the number of individuals or the number of tasks) is sufficiently large, a social choice function is virtually implementable in the *group* incentive case with *no* side payments if and only if such a function is exactly implementable in the individual incentive case with unbounded side payment devices. Thus, the class of implementable social choice functions is almost the *same* in both cases.

In order to prove this, we apply the concept of a *linking mechanism*, which was proposed by Jackson and Sonnenschein (2005). As in the case of the standard direct mechanism, the principal requires the group to make an announcement for each task about the observed private signal. The main difference between the linking mechanism and the direct mechanism is that the principal restricts the message space in advance by directing the group to ensure that the proportion of the tasks for which the group announces a private signal is approximately equal to the probability of this signal being observed for a single task. Since the total number of the tasks is sufficiently large, it is almost certain, based on the law of large numbers, that the realized proportion of the tasks for which each private signal is observed is almost the same as the probability of this signal being observed for a single task. Therefore, truth-telling, which induces the value of the social choice function for all tasks, is almost compatible with this message space restriction.

The essential finding of this paper is the clear resemblance between the functioning of this message space restriction and that of the incentive wage schemes in the standard approach. This resemblance can be elucidated using the following case in which the principal designs separate wage schemes for each individual in order to incentivize them to tell the truth. Let us suppose that an individual adopts a dishonest strategy that causes the frequency of announcing each signal to be different from the probability of this signal being observed. In such a case, a well-designed wage scheme can detect this dishonesty and the individual will be fined a large expected amount. In this sense, the functioning of the incentive wage scheme parallels that of message space restriction. On the other hand, if an individual adopts a dishonest strategy that causes the frequency of announcing each signal to be equal to the probability of this signal being observed, no wage scheme will detect this dishonesty. Therefore, we merely need to examine whether, in the absence of an incentive device, each individual has an incentive to adopt a dishonest strategy that causes the frequency of announcing each signal to be equal to the probability of this signal being observed. This implies that the necessary and sufficient condition for implementability is generally the same for both the individual incentive case with wage payment devices and the group incentive case with no such devices. Therefore, we can conclude that applying a linking mechanism is far more advantageous than designing an incentive wage scheme; this is because a linking mechanism enables us to avoid welfare distortions and positive information rents without narrowing the class of implementable social choice functions.

Needless to say, the arguments of this paper will be applied to a situation in which a principal contracts with a single individual and delegates multiple tasks to her. More importantly, we can extend our arguments to the case of multiple agents who and the principal are in conflict with each other. Jackson and Sonnenschein (2005) showed that the linking mechanism functions effectively with multiple agents, private values, and independent signals across the agents if the social choice function satisfies the *ex ante efficiency*. This paper characterizes the class of social choice functions that are virtually implemented by the linking mechanisms and presents an alternative sufficient condition, i.e., *supermodularity*. We do not require the private values assumption. We also investigate the correlated signals case.

In the agency literature in which a principal hires multiple workers, several studies such as Itoh (1993) and Baron and Besanko (1999) have demonstrated the superiority of group decisions over individual decisions. Baron and Besanko considered a setting in which each worker cannot verify her private signal to the other workers and, therefore, the workers, along with a neutral party, require a side payment contract in advance in order to be incentivized. In contrast, this paper assumes that each individual's private signal is verifiable by the other workers and not by the principal; therefore, the individuals in the group can share their private signals without suffering any loss.

We can find empirical evidences in which groups are used to elicit the private signals of individuals; these evidences are closely related to the basic idea of linking mechanisms. For instance, in developing countries, *community based targeting* has become popular; in this mechanism, governments or NGOs delegate to the community the authority to decide the recipients of the poverty reduction programs. Other targeting mechanisms that are based on reported household incomes or those that directly investigate household assets are also widely used; however, these mechanisms require substantial transaction costs or provide individuals with significant incentives to report incomes dishonestly (lower than the actual amount). Delegating to the community the authority to allocate aid can enhance the accuracy as well as the cost-effectiveness of targeting. See Coady, Grosch, and Hoddinott (2004) for more details.

In the economics theory literature, we find that some papers have presented concepts related to linking mechanisms even before the study by Jackson and Sonnenschein (2005). For instance, bundling goods by a monopolist (Armstrong (1999)), storable votes (Casella (2005) and Casella, Gelman, and Palfrey (2003)), and multimarket contact (Bernheim and Whinston (1990) and Matsushima (2001)). For more recent studies, see Eliaz, Ray, and Razin (2005) and Fang and Norman (2005a, 2005b).

It is extremely important to conduct laboratory experiments to show whether the linking mechanism functions effectively and the extent to what it does so. As Fehr and Falk (2002), Fehr and Gächter (2002), and Fehr, Gächter, and Kirschsteiger (1997) have shown through laboratory experiments, the incentive device of monetary rewards and punishments results in a decline in the reciprocal motives of real individuals. We conjecture that the incentive device of a linking mechanism is far more compatible with this reciprocal motive than is that of monetary rewards and punishments. As complementary research, we plan to conduct experiments on this subject.[4]

This paper is organized as follows. Section 2 presents an example that elucidates the basic concept discussed in this paper. Section 3 describes the single agent model. Section 4 presents the necessary condition for the virtual implementation of a social choice function. Section 5 introduces the linking mechanism and characterizes the class of social choice functions that it virtually implements. Section 6 characterizes the class of the social choice functions that are exactly implemented by inconstant wage schemes and shows the resemblance between the functioning of incentive wage schemes and that of the linking mechanism. Section 7 extends our results to the case of multiple agents.

[4] A recent study by Engelmann and Grimm (2006) presents experimental research on linking mechanisms. They reported that linking mechanisms function effectively in laboratories. Moreover, the experiments conducted by Casella, Gelman, and Palfrey (2003) on storable votes closely related to linking mechanisms reported that the storable votes performed very well.

2. Example

The following example will help in understanding the concepts presented in this paper. Consider a situation in which a principal hires K individuals and delegates each individual $h \in \{1,...,K\}$ the h-th task. Each individual h observes a private signal ω_h that is randomly determined to be either 0 or 1 with probability $\frac{1}{2}$. These private signals are drawn independently. The principal cannot observe these private signals; therefore, each individual is required to announce the signal between 0 and 1 that she actually observes. After the announcements, the principal compels each individual h to make an alternative choice a_h, which is either 0 or 1, and pays a *nonnegative* wage $w_h \geq 0$ according to a contract specified in advance. If individual h observes the private signal $\omega_h \in \{0,1\}$, makes an alternative choice $a_h \in \{0,1\}$, and receives wage $w_h \geq 0$, this individual's payoff is $w_h - a_h(\omega_h + 1)$, and the principal obtains a net profit $\frac{3}{2}a_h - w_h$ from the h-th task. The principal guarantees each individual an ex ante expected payoff that is either equal to or greater than the outside opportunity $-\frac{1}{2}$.

If the private signals are verifiable and there is no adverse selection problem, the principal can achieve the first-best allocation such that for every $h \in \{1,...,K\}$,

$$a_h = 0 \text{ and } w_h = 0 \text{ if } \omega_h = 1$$

and

$$a_h = 1 \text{ and } w_h = 0 \text{ if } \omega_h = 0.$$

Therefore, the principal extracts the full surplus $\frac{3}{4}$ as her ex ante expected payoff for each task. However, if the private signals are not verifiable, the principal must incentivize the individuals to announce their true private signals. In order to obtain the first-best alternative choices, the principal must pay to each individual h a positive wage $w_h = 1$ whenever $\omega_h = 0$. This, along with the nonnegativity of the wages, implies that the individual earns a positive information rent $\frac{1}{2}$ and, therefore, the principal's expected payoff for each task $h \in \{1,...,K\}$ is approximately $\frac{1}{4}$, which is much lesser than the full surplus $\frac{3}{4}$.

In order to overcome the failure to extract the full surplus, the principal will compel the individuals to establish a *working group* in the following manner. The individuals agree to share their private signals and coordinate their announcements in order to maximize the average of their payoffs. The principal regards this group as a single agent with whom she makes a contract termed as the linking mechanism—the basic idea of this mechanism was proposed by Jackson and Sonnenschein (2005). According to the linking mechanism, the principal's wage payments are fixed at zero, irrespective of the group's announcements. Moreover, the principal requires the group to announce the signal 0 for at least $\frac{K}{2}$ tasks, irrespective of the observations of the individuals' private signals. In cases where K is sufficiently large, by restricting the group's possible announcements through this mechanism, the principal succeeds in incentivizing the group to tell the truth in any manner possible. In fact, based on the law of large numbers, the principal can virtually achieve the first-best allocation without paying any positive wages, i.e., she can achieve the full surplus extraction.

3. The Model

This paper investigates the following situation in which a principal delegates K distinct tasks to a *single* agent, i.e., the agent is required to choose a profile of K alternatives $a^K = (a_1,...,a_k) \in A^K \equiv \times_{h=1}^K A_h$, where for each $h \in \{1,...,K\}$, A_h and a_h denote the set of alternatives and an alternative for the h-th task, respectively. The agent observes a profile of K private signals $\omega^K = (\omega_1,...,\omega_k) \in \Omega^K \equiv \times_{h=1}^k \Omega_h$, where $\omega_h \in \Omega_h$ is the signal for the h-th task. This paper focuses on *symmetric* models in the sense that for every $h \in \{1,...,K\}$,

$$A_h = A \text{ and } \Omega_h = \Omega.$$

Let $\sharp\Omega = I < \infty$. The agent observes a profile of private signals $\omega^K \in \Omega^K$ with positive probability $p^K(\omega^K) > 0$. We assume *independence* in the sense that there exists a probability function $p : \Omega \to [0,1]$ such that

$$p^K(\omega^K) = \Pi_{h=1}^K p(\omega_h) \text{ for all } \omega^K \in \Omega^K.$$

The principal is unaware of the private signal profile $\omega^K \in \Omega^K$ and therefore requires the agent to announce a message on the basis of the mechanism given by $\Gamma(K) \equiv (M,(g^K,t^K))$. Here, M is the nonempty finite set of messages for the agent, $g^K : M \to \Delta(A^K)$,[5] and $t^K = (t_h)_{h=1}^K : M \to R^K$. When the agent announces a message $m \in M$, the principal compels her to choose any profile of alternatives $a^K \in A^k$ with probability $g^K(m)[a^K]$[6]; the principal herself chooses the K profile of side payments given by $w^K = t^K(m) = (t_1(m),...,t_K(m)) \in R^K$. Here, for every $h \in \{1,...,K\}$, $w_h \equiv t_h(m) \in R$ is regarded as the wage payment for the h-th task. When the agent observes $\omega^K \in \Omega^K$ and chooses $a^K \in A^K$ and the principal chooses $w^K \in R^K$, the agent's payoff is given by $v^K(a^K,w^K,\omega^K) \in R$. We assume *expected utility*. For every simple lottery α over A^K, let $v^K(\alpha,w^K,\omega^K) \equiv \sum_{a^K \in \Lambda} v^K(a^K,w^K,\omega^K)\alpha(a^K)$, where Λ denotes the support of α.

[5] For every set Φ, the set of simple lotteries over Φ is denoted by $\Delta(\Phi)$.

[6] In order to focus on the adverse selection problem, we assume that the probabilistic alternative choices are verifiable by the court.

This paper assumes *additive separability* in the sense that

$$v^K(a^K, w^K, \omega^K) = \frac{1}{K}\left\{\sum_{h=1}^{K} u(a_h, \omega_h) + d^K(w^K)\right\}.$$

Here, $d^K : R^K \to R$ is increasing and satisfies *symmetry* in the sense that for every $w^K \in R^K$ and every $\tilde{w}^K \in R^K$, if there exists a permutation on $\{1,...,K\}$, $\pi : \{1,...,K\} \to \{1,...,K\}$ such that

$$w_h = \tilde{w}_{\pi(h)} \text{ for all } h \in \{1,...,K\},$$

then

$$d^K(w^K) = d^K(\tilde{w}^K).$$

The class of models in this paper includes the following two cases.

Case I: The agent is regarded as a *single* individual. There exists an increasing and continuous function $d : R \to R$ such that

$$d^K(w^K) = d(\sum_{h=1}^{K} w_h) \text{ for all } w^K \in R^K,$$

i.e., the individuals' payoff depends on the sum of the side payments.

Case II: The agent is regarded as a *group* comprising K individuals. There exists an increasing and continuous function $e : R \to R$ such that

$$d^K(w^K) = \sum_{h=1}^{K} e(w_h) \text{ for all } w^K \in R^K.$$

Each individual $h \in \{1,...,K\}$ performs the h-th task and her corresponding payoff is given by $u(a_h, \omega_h) + e(w_h)$. The K individuals agree to maximize the average of their payoffs $\frac{1}{K}\left\{\sum_{h=1}^{K}\{u(a_h, \omega_h) + e(w_h)\}\right\}$.

An important intersection between Cases I and II is the risk-neutral agent case, where $d^K(w^K) = \sum_{h=1}^{K} w_h$.

A *strategy* (for the agent) is defined as the function $\sigma = \sigma^K : \Omega^K \to M$. We denote the set of strategies by $\Sigma = \Sigma^K$. A strategy $\sigma \in \Sigma$ is said to be a *best response* in $\Gamma(K)$ if for every $\sigma' \in \Sigma$,

$$E\left[v^K(g^K(m), t^K(m), \omega^K) \middle| \Gamma(K), \sigma\right] \geq E\left[v^K(g^K(m), t^K(m), \omega^K) \middle| \Gamma(K), \sigma'\right].$$

Here, $E\left[\cdot\middle|\Gamma(K),\sigma\right]$ is the expectation operator over $\omega^K \in \Omega^K$ given $(\Gamma(K),\sigma)$. A social choice function is defined as $f:\Omega \to A$. Irrespective of $h \in \{1,...,K\}$, $f(\omega) \in A$ is regarded as the socially desirable alternative choice for the h-th task when the agent observes $\omega_h = \omega$. A social choice function f is said to be *exactly implementable with respect to* K if there exist a mechanism $\Gamma(K)$ and a best response $\sigma \in \Sigma$ in $\Gamma(K)$ such that

(1) $$g^K(\sigma(\omega^K))[f^K(\omega^K)] = 1 \text{ for all } \omega^K \in \Omega^K,$$

where we denote $f^K(\omega^K) \equiv (f(\omega_1),..., f(\omega_K))$. An infinite sequence of mechanisms $(\Gamma(K))_{K=1}^{\infty}$ is said to *virtually implement a social choice function* f if for every $\eta > 0$, there exists $\bar{K}$ such that for every $K \geq \bar{K}$, there is a best response $\sigma \in \Sigma$ in $\Gamma(K)$ that satisfies

(2) $$E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid a_h = f(\omega_h)\}}{K}\middle|\Gamma(K),\sigma\right] \geq 1-\eta .$$

4. Necessary Condition for Virtual Implementation

This section shows that the following condition is necessary for *virtual* implementation.

Condition 1: For every $L \in \{2,\cdots,I\}$ and every $(\omega(1),\cdots,\omega(L)) \in \Omega^L$, if

$$\omega(l) \neq \omega(l') \text{ for all } l \in \{1,\ldots,L\} \text{ and } l' \in \{1,\ldots,L\} \setminus \{l\},$$

then

$$(3) \qquad \sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) \geq \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l)),$$

where we denote $L+1=1$.

Condition 1 implies that if an agent lies according to any permutation with regard to Ω with the same probability across all the possible signals, her payoff never improves.

Theorem 1: *If there exists an infinite sequence of mechanisms* $(\Gamma(K))_{K=1}^{\infty}$ *that virtually implement a social choice function* f*, Condition 1 holds.*

Proof: Suppose that Condition 1 does not hold, i.e., there exists $L \in \{2,\cdots,I\}$ and $(\omega(1),\cdots,\omega(L))$ such that $\omega(l) \neq \omega(l')$ for all $l \in \{1,\ldots,L\}$ and $l' \in \{1,\ldots,L\} \setminus \{l\}$, and

$$\sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) < \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l)).$$

Further, suppose that $(\Gamma(K))_{K=1}^{\infty}$ virtually implements f with respect to p. By the revelation principle,[7] we can assume that for every K, $\Gamma(K) = (M, g^K, t^K)$ is a *direct* mechanism where $M = \Omega^K$; the *truthful* strategy $\hat{\sigma} = \hat{\sigma}^K$, which is defined by $\hat{\sigma}^K(\omega^K) = \omega^K$ for all $\omega^K \in \Omega^K$, is a best response in $\Gamma(K)$; and

$$(4) \qquad \lim_{K\to\infty} E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid a_h = f(\omega_h)\}}{K} \middle| \Gamma(K), \hat{\sigma}^K\right] = 1.$$

Let $M = \times_{h=1}^{K} M_h$, $M_h = \Omega$, $m = (m_1,\cdots,m_K) \in M$, and

[7] See Myerson (1979) and Fudenberg and Tirole (1993, Chapter 7).

$$\sigma(\omega^K) = (\sigma_1(\omega^K), \cdots, \sigma_K(\omega^K)),$$

where $\sigma_h(\omega^K) \in \Omega$ for all $h \in \{1,...,K\}$. Without loss of generality, we can assume that for every K, $\Gamma(K) = (M, g^K, t^K)$ is *symmetric* in the sense that for every $m \in M$, every $a^K \in A^K$, and every permutation $\pi : \{1,\cdots,K\} \to \{1,\cdots,K\}$,

$$g^K(m)[a^K] = g^K(m^\pi)[a^{K,\pi}],$$

where $m^\pi = (m_1^\pi,...,m_K^\pi) \in M$ and $a^{K,\pi} = (a_1^{K,\pi},...,a_K^{K,\pi}) \in A^K$ are defined by

$$m_{\pi(h)}^\pi = m_h \text{ and } a_{\pi(h)}^{K,\pi} = a_h \text{ for all } h \in \{1,\cdots,K\}.$$

See Appendix A for the proof that it is sufficient to consider symmetric mechanisms.

For any $\lambda > 0$, let

$$\Omega^{K*}(\lambda) \equiv \left\{ \omega^K \in \Omega^K \mid \left| \frac{\sharp\{h \in \{1,\cdots,K\} \mid \omega_h = \omega\}}{K} - p(\omega) \right| < \lambda \ \textit{for all}\ \omega \in \Omega \right\}.$$

The law of large numbers implies that for all $\lambda > 0$,

$$\lim_{K\to\infty} \sum_{\omega^K \in \Omega^{K*}(\lambda)} p^K(\omega^K) = 1.$$

Therefore, there is an infinite sequence of positive real numbers $(\lambda_K)_{K=1}^\infty$ such that $\lim_{K\to\infty} \lambda_K = 0$ and

(5) $$\lim_{K\to\infty} \sum_{\omega^K \in \Omega^{K*}(\lambda_K)} p^K(\omega^K) = 1.$$

We assume a sufficiently large K. From (4) and (5), it follows that there exists $\tilde{\omega}^K = (\tilde{\omega}_1,...,\tilde{\omega}_K) \in \Omega^{K*}(\lambda_K)$ such that

(6) $g^K(\tilde{\omega}^K)[f^K(\tilde{\omega}^K)]$ is close to 1.

We specify a strategy $\sigma' \in \Sigma$ as follows.

(i) For every $l \in \{1,\cdots,L\}$, the number of $h \in \{1,\cdots,K\}$ satisfying $\tilde{\omega}_h = \omega(l)$ and $\sigma_h'(\tilde{\omega}^K) = \omega(l+1)$ is equal to

$$\min_{l\in\{1,\cdots,L\}} \sharp\{h \in \{1,\cdots,K\} \mid \tilde{\omega}_h = \omega(l)\}.$$

(ii) For every $h \in \{1,\cdots,K\}$, either $\sigma_h'(\tilde{\omega}^K) = \tilde{\omega}_h$ or

$$\tilde{\omega}_h = \omega(l) \text{ and } \sigma'_h(\tilde{\omega}^K) = \omega(l+1) \text{ for some } l \in \{1, \cdots, L\}.$$

(iii) For every $\omega^K \in \Omega^K \setminus \{\tilde{\omega}^K\}$,

$$\sigma'(\omega^K) = \sigma(\omega^K).$$

From the definition of σ', it follows that there exists a permutation π on $\{1, ..., K\}$ such that $\sigma'(\tilde{\omega}^K) = \tilde{\omega}^{K,\pi}$,

(7) $$\sum_{h=1}^{K} u(f(\tilde{\omega}_h), \tilde{\omega}_h) < \sum_{h=1}^{K} u(f(\tilde{\omega}_{\pi(h)}), \tilde{\omega}_h),$$

and

(8) $$g^K(\sigma'(\tilde{\omega}^K))[f^K(\tilde{\omega}^{K,\pi})] \text{ is close to 1,}$$

where $\tilde{\omega}^{K,\pi} = (\tilde{\omega}_1^{\pi}, ..., \tilde{\omega}_K^{\pi}) \in \Omega^K$ and $\tilde{\omega}_{\pi(h)}^{\pi} = \tilde{\omega}_h$ for all $h \in \{1, \cdots, K\}$. From (6), (7), and (8), it follows that

$$E\left[\sum_{h=1}^{K} u(a_h, \tilde{\omega}_h) \middle| \Gamma(K), \hat{\sigma}, \tilde{\omega}^K\right] < E\left[\sum_{h=1}^{K} u(a_h, \tilde{\omega}_h) \middle| \Gamma(K), \sigma', \tilde{\omega}^K\right].$$

Since $\Gamma(K)$ is symmetric, it follows that

$$E\left[\sum_{h=1}^{K} d(w_h) \middle| \Gamma(K), \hat{\sigma}, \tilde{\omega}^K\right] = E\left[\sum_{h=1}^{K} d(w_h) \middle| \Gamma(K), \sigma', \tilde{\omega}^K\right].$$

Therefore,

$$v^K(g^K(\hat{\sigma}(\tilde{\omega}^K)), t^K(\hat{\sigma}(\tilde{\omega}^K)), \tilde{\omega}^K) < v^K(g^K(\sigma'(\tilde{\omega}^K)), t^K(\sigma'(\tilde{\omega}^K)), \tilde{\omega}^K).$$

This contradicts the fact that $\hat{\sigma}$ is a best response in $\Gamma(K)$.

Q.E.D.

5. Linking Mechanisms

This section shows that Condition 1 is also sufficient for virtual implementation. The proof of this statement is constructive. Based on Jackson and Sonnenschein (2005), we define the linking mechanism $\Gamma^*(K;f)=\Gamma^*(K)=(M,(g^K,t^K))$ as follows. We specify $B(\cdot,K):\Omega\to\{0,...,K\}$ such that

$$\sum_{\omega\in\Omega} B(\omega,K)=K\,.$$

Moreover, for every $b:\Omega\to\{0,...,K\}$ such that $\sum_{\omega\in\Omega} b(\omega)=K$,

$$\sum_{\omega\in\Omega}\left|\frac{B(\omega,K)}{K}-p(\omega)\right|\le\sum_{\omega\in\Omega}\left|\frac{b(\omega)}{K}-p(\omega)\right|. \tag{9}$$

The definition of $B(\cdot,K)$ implies that $\frac{B(\omega,K)}{K}$ approximates $p(\omega)$ for a sufficiently large K, i.e.,

$$\lim_{K\to\infty}\frac{B(\omega,K)}{K}=p(\omega)\quad\text{for all }\ \omega\in\Omega\,. \tag{10}$$

Let

$$M_h=\Omega\quad\text{for all}\quad h\in\{1,...,K\}\,,$$

$$M=\left\{m\in\Omega^K\,\middle|\,\sharp\{h\in\{1,\cdots,K\}\mid m_h=\omega\}=B(\omega,K)\ \textit{for all}\ \omega\in\Omega\right\}, \tag{11}$$

and

$$g^K(m)[f^K(m)]=1\quad\text{for all}\quad m\in M\,.$$

Moreover, there exists a real number $z\in R$ such that

$$t^K(m)=(z,...,z)\quad\text{for all}\quad m\in M\,.$$ [8]

For convenience of our argument, without loss of generality, we assume $z=0$. The agent has to announce each $\omega\in\Omega$ exactly $B(\omega,K)$ times. This along with (10) implies that for a sufficiently large K, the proportion of the tasks for which the agent announces ω is almost the same as the probability $p(\omega)$ of ω occurring. In the linking mechanism, the side payments are constant across the agent's possible

[8] Since the side payments are constant in the linking mechanism, we can weaken the assumption of additive separability in this section by requiring instead that there exists $u:A\times\Omega\to R$ such that $v^K(a^K,w^K,\omega^K)=\frac{1}{K}\sum_{h=1}^{K}u(a_h,\omega_h)$ for $w^K=(z,...,z)$.

announcements, i.e., the agent receives a constant wage $z=0$ for each task, irrespective of the observation of the private signal profile. This implies that the linking mechanism is free from the welfare loss that occurs due to the agent's risk aversion, and it is also free from the failure of the principal to extract the full surplus owing to the agent's positive information rent that is caused by the nonnegative wage payments. Moreover, the linking mechanism is regarded as being well behaved from the practical viewpoint because it does not depend on the finer details of the agent's utility function u.[9]

The following theorem shows that Condition 1 is sufficient for the linking mechanisms to virtually implement the social choice function.

Theorem 2: *Under Condition 1, a social choice function* f *is virtually implemented by the infinite sequence of the linking mechanisms* $(\Gamma^*(K))_{K=1}^{\infty}$.

Proof: Suppose that there exists $\eta>0$ such that for every $\bar{K}$, there exists $K\geq\bar{K}$ that satisfies the condition that for every best response $\sigma\in\Sigma$ in $\Gamma^*(K)$,

$$E\left[\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,a_h\neq f(\omega_h)\}}{K}\middle|\Gamma^*(K),\sigma\right]>\eta .$$

As in the proof of Theorem 1, we can choose an infinite sequence of positive real numbers $(\lambda_K)_{K=1}^{\infty}$ satisfying (5) and $\lim_{K\to\infty}\lambda_K=0$. From (5) and (10), it follows that for every sufficiently large K, every $\omega^K\in\Omega^{K*}(\lambda_K)$, and every $\omega\in\Omega$,

(12) $$\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\omega_h=\omega\}}{K}\text{ is approximated by }\frac{B(\omega,K)}{K}.$$

Consider a best response $\sigma\in\Sigma$ in $\Gamma^*(K)$ satisfying the condition that for every best response $\tilde{\sigma}\in\Sigma$ in $\Gamma^*(K)$,

(13) $$E\left[\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\omega^K)\neq\omega_h\}}{K}\middle|\Gamma^*(K),\sigma\right]$$

$$\leq E\left[\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\tilde{\sigma}_h(\omega^K)\neq\omega_h\}}{K}\middle|\Gamma^*(K),\tilde{\sigma}\right].$$

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[9] The linking mechanism, however, depends on the probability function p as well as the social choice function f.

It is evident that

$$(14)\qquad E\left[\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,a_h\neq f(\omega_h)\}}{K}\middle|\Gamma^*(K),\sigma\right]>\eta\,.$$

The left-hand side of (14) is rewritten as

$$\sum_{\omega^K\in\Omega^K} p^K(\omega^K)\times\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\omega^K)\neq\omega_h\}}{K}$$
$$=\sum_{\omega^K\in\Omega^{K*}(\lambda_K)} p^K(\omega^K)\times\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\omega^K)\neq\omega_h\}}{K}$$
$$+\sum_{\omega^K\in\Omega\backslash\Omega^{K*}(\lambda_K)} p^K(\omega^K)\times\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\omega^K)\neq\omega_h\}}{K}\,.$$

For every sufficiently large K, the last term is close to zero; therefore, the left-hand side of (14) is approximated by

$$\sum_{\omega^K\in\Omega^{K*}(\lambda_K)} p^k(\omega^K)\times\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\omega^K)\neq\omega_h\}}{K}\,.$$

This implies that there exists $\hat{\omega}^K\in\Omega^{K*}(\lambda_K)$ such that

$$(15)\qquad \frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\hat{\omega}^K)\neq\hat{\omega}_h\}}{K}>\eta\,.$$

A strategy $\sigma\in\Sigma$ is said to be *cyclic* for $\omega^K\in\Omega^K$ if there exist $S\subseteq\{1,\cdots,K\}$ and a one-to-one function $\tau:\{1,...,\#S\}\to S$ such that $2\leq\sharp S\leq K$,

$$\omega_s\neq\omega_{s'}\ \text{ for all }\ s\in S\ \text{ and }\ s'\in S\setminus\{s\},$$

and for every $l\in\{1,...,\#S\}$,

$$\sigma_{\tau(l)}(\omega^K)=\omega_{\tau(l+1)},\ \text{where}\ \sharp S+1=1.$$

Note that if σ is not cyclic for $\hat{\omega}^K$, the proportion of the tasks for which the agent announces incorrect private signals, i.e., $\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\sigma_h(\hat{\omega}^K)\neq\hat{\omega}_h\}}{K}$, is less than or equal to

$$(16)\qquad \sum_{\omega\in\Omega}\left|\frac{\sharp\{h\in\{1,\cdots,K\}\,|\,\hat{\omega}_h=\omega\}}{K}-\frac{B(\omega,K)}{K}\right|,$$

which is close to zero because of (12). This, however, contradicts (14). Hence, σ must be cyclic for $\hat{\omega}^K$, i.e., there exist $S\subseteq\{1,\cdots,K\}$ and a one-to-one function $\tau:\{1,...,\#S\}\to S$ such that $2\leq\sharp S\leq K$,

$$\hat{\omega}_s\neq\hat{\omega}_{s'}\ \text{ for all }\ s\in S\ \text{ and }\ s'\in S\setminus\{s\},$$

and

$$\sigma_{\tau(l)}(\hat{\omega}^K) = \hat{\omega}_{\tau(l+1)} \text{ for all } l \in \{1,...,\# S\}.$$

We specify a strategy $\tilde{\sigma} \in \Sigma$ by

$$\tilde{\sigma}_s(\hat{\omega}^K) = \hat{\omega}_s \text{ for all } s \in S,$$

$$\tilde{\sigma}_s(\hat{\omega}^K) = \sigma_s(\hat{\omega}^K) \text{ for all } s \notin S,$$

and

$$\tilde{\sigma}(\omega^K) = \sigma(\omega^K) \text{ for all } \omega^K \in \Omega^K \setminus \{\hat{\omega}^K\}.$$

From the definition of $\tilde{\sigma}$, it follows that the expected number of the tasks for which the agent lies according to $\tilde{\sigma}$ is less than that according to σ, i.e.,

$$E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid \tilde{\sigma}_h(\omega^K) \neq \omega_h\}}{K} \middle| \Gamma^*(K), \tilde{\sigma}\right]$$

$$< E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid \sigma_h(\omega^K) \neq \omega_h\}}{K} \middle| \Gamma^*(K), \sigma\right].$$

From Condition 1 and the fact that σ is a best response in $\Gamma^*(K)$, it follows that $\tilde{\sigma}$ is another best response in $\Gamma^*(K)$. This, however, contradicts (13).

Q.E.D.

Theorems 1 and 2 imply that Condition 1 is necessary and sufficient for virtual implementation. These theorems also imply that whenever a sufficiently large number of tasks are delegated to the agent, i.e., K is sufficiently large, all that is required for virtual implementation is to check whether the linking mechanism functions or not.

The following proposition shows that we can replace Condition 1 with a more intuitive condition termed as *supermodularity*.[10]

Condition 2 (Supermodularity): Ω is an ordered set with $\geq$, and for every $\omega, \omega', \omega'', \omega''' \in \Omega$,

$$u(f(\omega), \omega') + u(f(\omega''), \omega''')$$
$$\leq u(f(\omega \vee \omega''), \omega' \vee \omega''') + u(f(\omega \wedge \omega''), \omega' \wedge \omega'''),$$

where $\omega \vee \omega'' = \max\{\omega, \omega''\}$ and $\omega' \wedge \omega''' = \min\{\omega', \omega'''\}$.

[10] See Topkis (1979) and Fudenberg and Tirole (1993, Chapter 12) for supermodularity and its related concepts.

Proposition 3: *Condition 2 implies Condition 1.*

Proof: Consider any $L \in \{2, \cdots, I\}$ and $(\omega(1), \cdots, \omega(L)) \in \Omega^L$ such that

$$\omega(l) \leq \omega(l+1) \text{ for all } l \in \{1, \cdots, L-1\}.$$

From Condition 2, it follows that the right-hand side of (3) is rewritten as

$$u(f(\omega(2)), \omega(1)) + u(f(\omega(1)), \omega(L)) + \sum_{l=2}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$\leq u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(L)) + \sum_{l=2}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$= u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(L)) + u(f(\omega(3)), \omega(2))$$

$$+ \sum_{l=3}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$\leq u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(2)) + u(f(\omega(3)), \omega(L))$$

$$+ \sum_{l=3}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$\vdots$$

$$\leq \sum_{l=1}^{L} u(f(\omega(l)), \omega(l)),$$

which implies Condition 1.

Q.E.D.

From Theorem 2 and Proposition 3, it follows that supermodularity is sufficient for the linking mechanisms to virtually implement the social choice function.

6. Exact Implementation

This section focuses on Case II in which K individuals exist and the principal delegates the h-th task to each individual $h \in \{1,...,K\}$. Individual $h's$ payoff is given by $u(a_h, \omega_h) + e(w_h)$ for all $w^K \in R^K$, where $e: R \to R$ is increasing and continuous. This section assumes that e is *unbounded* in that for every positive real number $B > 0$, there exists a positive real number $w > 0$ such that

$$e(w) \geq B \text{ and } e(-w) \leq -B.$$

Evidently, this assumption holds when the individuals are risk neutral in that $e(w) = w$ for all $w \in R$. The agent is regarded as the group that comprises these K individuals.

This section investigates *exact* implementation that requires the value of a social choice function to be realized with certainty, irrespective of which private signal profile the agent observes. The following proposition shows that Condition 1 is necessary and sufficient for exact implementation, irrespective of K; therefore, the necessary and sufficient condition is the same for both virtual and exact implementation.

Proposition 4: *A social choice function f is exactly implementable with respect to K if and only if Condition 1 holds.*

Proof: We can apply Theorem 1 proposed by Fan (1956) in the same manner as it was used in D'Aspremont and Gèrard-Varet (1979, Theorem 7). For the complete proof, see Appendix B.

In contrast with virtual implementation, in order to exactly implement a nontrivial social choice function, we have to design a mechanism $\Gamma(K) = (M, (g^K, t^K))$ that conditions the side payments $t^K(m)$ on the agent's announcements m. If we confine our analysis to mechanisms in which $t^K(m)$ is constant across the agent's possible announcements m, then the exactly implementable social choice functions are the only trivial ones, i.e., any exactly implementable social choice function f must satisfy

$$u(f(\omega), \omega) \geq u(f(\omega'), \omega) \text{ for all } \omega \in \Omega \text{ and all } \omega' \in \Omega.$$

These inequalities imply that either there is no conflict of interest between the principal and the agent or the social choice function does not take into account the principal's welfare.

Condition 1, which is necessary and sufficient for exact implementation based on Proposition 4, does not depend on K. Therefore, the set of exactly implementable social choice functions is the same for both the situation in which the K individuals agree to maximize the average of their payoffs and the one in which they do not agree and make decisions in their own self-interest.

From Theorem 1 presented in Fan (1956), it follows that a necessary and sufficient condition for the existence of such an r is that for every $\mu : \Omega^2 \to R_+ \bigcup \{0\}$, if

$$\sum_{\tilde{\omega} \neq \omega \in \Omega} \left\{ \mu(\omega, \tilde{\omega}) - \mu(\tilde{\omega}, \omega) \right\} = 0 \quad \text{for all} \quad \omega \in \Omega ,$$

then

$$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \neq \omega \in \Omega} \left\{ u(f(\omega), \omega) - u(f(\tilde{\omega}), \omega) \right\} \mu(\omega, \tilde{\omega}) \geq 0 .$$

Without loss of generality, we can focus only on the set of functions $\mu : \Omega^2 \to R_+ \bigcup \{0\}$ such that $\sum_{\omega' \in \Omega} \mu(\omega, \omega') = 1$ for all $\omega \in \Omega$, i.e., the set of mixed strategies in the direct mechanism, where $\mu(\omega, \omega')$ is the probability that the agent announces ω' given that she observes ω. The above condition is equivalent to the condition that for every mixed strategy μ in the direct mechanism, if the frequencies of announcing private signals are the same as the probabilities of these signals being observed, i.e.,

$$p(\omega' \mid \mu) \equiv \sum_{\omega \in \Omega} \mu(\omega, \omega') p(\omega) = p(\omega') \quad \text{for all} \quad \omega' \in \Omega ,$$

the ex ante expected payoff with no side payments induced by the dishonest mixed strategy is not greater than that induced by the honest strategy, i.e.,

$$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \in \Omega} u(f(\tilde{\omega}), \omega) \mu(\omega, \tilde{\omega}) p(\omega) \leq \sum_{\omega \in \Omega} u(f(\omega), \omega) p(\omega) .$$

Based on the law of large numbers, this inequality implies that the requirement of truth-telling being a best response is almost satisfied when the number of the tasks K is sufficiently large. Hence, the functioning of the message space restriction in the linking mechanism for the group parallels that of the incentive payment scheme in the direct mechanism for each individual.

In the case wherein the individuals make decisions in their own interest and the principal designs independent mechanisms that incentivize each individual separately, every individual has to accept the risk that results from inconstant payments. This will result in significant welfare distortion when the individuals are risk averse and the principal is risk neutral. In contrast, in the case wherein the individuals agree to maximize the average of their payoffs, the principal can apply the linking mechanisms to virtually implement the social choice function successfully. In this case, we do not require any incentive payment device and, therefore, there is no welfare distortion due to risk sharing.

7. Multiple Agents

This section generalizes the previous results to a case in which *multiple* n agents who observe their respective private signal profile are in conflict with each other over their own interests.

7.1. The Model

Let $\Omega = \times_{i=1}^{n} \Omega_{[i]}$ and $\omega = (\omega_{[1]}, ..., \omega_{[n]}) \in \Omega$. Each agent $i \in \{1, ..., n\}$ observes a profile of K private signals $\omega_{[i]}^{K} = (\omega_{[i],1}, ..., \omega_{[i],K}) \in \Omega_{[i]}^{K}$, where $\omega_{[i],h} \in \Omega_{[i]}$ is the private signal for the h-th task that agent i observes. A mechanism is defined as $\Gamma(K) \equiv (M, (g^K, t^K))$, where $M = \times_{i=1}^{n} M_{[i]}$, $M_{[i]}$ is the nonempty finite set of messages for agent i, $t^K = (t_{[i]}^{K})_{i=1}^{n}$, and $t_{[i]}^{K} = (t_{[i],h})_{h=1}^{K} : M \to R^K$. When the agents observe $\omega^K \in \Omega^K$ and choose $a^K \in A^K$ and the principal chooses $w^K = (w_{[i]}^{K})_{i=1}^{n}$, agent $i's$ payoff is

$$v_{[i]}^{K}(a^K, w_{[i]}^{K}, \omega^K) = \frac{1}{K}\left\{ \sum_{h=1}^{K} u_{[i]}(a_h, \omega_h) + d_{[i]}^{K}(w_{[i]}^{K}) \right\},$$

where $d_{[i]}^{K} : R^K \to R$ is increasing and satisfies symmetry. A *strategy* for agent i is defined as $\sigma_{[i]} = \sigma_{[i]}^{K} : \Omega_{[i]}^{K} \to M_{[i]}$. Let $\Sigma_{[i]} = \Sigma_{[i]}^{K}$ denote the set of strategies for agent i. Let $\Sigma = \prod_{i=1}^{n} \Sigma_{[i]}$. A strategy profile $\sigma \in \Sigma$ is said to be a *Nash equilibrium* in $\Gamma(K)$ if for every $i \in \{1, ..., n\}$ and every $\sigma'_{[i]} \in \Sigma_{[i]}$,

$$E\left[v_{[i]}^{K}(g^K(m), t_{[i]}^{K}(m), \omega^K) \middle| \Gamma(K), \sigma \right]$$

$$\geq E\left[v_{[i]}^K(g^K(m),t_{[i]}^K(m),\omega^K)\middle|\Gamma(K),(\sigma'_{[i]},\sigma_{-[i]})\right].$$

A social choice function f is said to be *exactly implementable with respect to* K if there exists a mechanism $\Gamma(K)$ and a Nash equilibrium $\sigma \in \Sigma$ in $\Gamma(K)$ such that

$$g^K(\sigma(\omega^K))[f^K(\omega^K)] = 1 \text{ for all } \omega^K \in \Omega^K.$$

For each $\varepsilon > 0$, a strategy profile $\sigma \in \Sigma$ is said to be a $\varepsilon-$*Nash equilibrium* in $\Gamma(K)$ if for every $i \in \{1,...,n\}$ and every $\sigma'_{[i]} \in \Sigma_{[i]}$,

$$E\left[v_{[i]}^K(g^K(m),t_{[i]}^K(m),\omega^K)\middle|\Gamma(K),\sigma\right] + \varepsilon$$

$$\geq E\left[v_{[i]}^K(g^K(m),t_{[i]}^K(m),\omega^K)\middle|\Gamma(K),(\sigma'_{[i]},\sigma_{-[i]})\right].$$

An infinite sequence of mechanisms $(\Gamma(K))_{K=1}^{\infty}$ is said to *virtually implement a social choice function* f if for every $\eta > 0$ and every $\varepsilon > 0$, there exists $\bar{K}$ such that for every $K \geq \bar{K}$, there is a $\varepsilon-$Nash equilibrium $\sigma \in \Sigma$ in $\Gamma(K)$ satisfying

$$E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid a_h = f(\omega_h)\}}{K}\middle|\Gamma(K),\sigma\right] \geq 1-\eta.$$

Let

$$p_{[i]}(\omega_{[i]}) \equiv \sum_{\omega_{-[i]}\in\Omega_{-[i]}} p(\omega).$$

We define the linking mechanism $\Gamma^*(K;f) = \Gamma^*(K) = (M,(g^K,t^K))$ in ways that

$$g^K(m)[f^K(m)] = 1 \text{ for all } m \in M,$$

for every $i \in \{1,...,n\}$,

$$M_{[i],h} = \Omega \text{ for all } h \in \{1,...,K\},$$

$$M_{[i]} = \left\{m_{[i]} \in \Omega_{[i]}^K \middle| \sharp\left\{h \in \{1,\cdots,K\} \mid m_{[i],h} = \omega_{[i]}\right\} = B_i(\omega_{[i]},K) \textit{ for all } \omega_{[i]} \in \Omega_{[i]}\right\},$$

and

$$t_{[i],h}^K(m) = 0 \text{ for all } h \in \{1,...,K\} \text{ and all } m \in M.$$

Here, $B_i(\cdot,K) : \Omega_{[i]} \to \{0,...,K\}$ is specified as satisfying $\sum_{\omega_{[i]}\in\Omega_{[i]}} B_i(\omega_{[i]},K) = K$.

Moreover, for every $b_{[i]} : \Omega_{[i]} \to \{0,...,K\}$ such that $\sum_{\omega_{[i]}\in\Omega_{[i]}} b_{[i]}(\omega_{[i]}) = K$,

$$\sum_{\omega_{[i]}\in\Omega_{[i]}}\left|\frac{B_{[i]}(\omega_{[i]},K)}{K}-p_{[i]}(\omega_{[i]})\right|\leq\sum_{\omega_{[i]}\in\Omega_{[i]}}\left|\frac{b_{[i]}(\omega_{[i]})}{K}-p_{[i]}(\omega_{[i]})\right|.$$

The following conditions are direct extensions of Conditions 1 and 2, respectively.

Condition 3: For every $i\in\{1,...,n\}$, every $L\in\{2,\cdots,I\}$, and every $(\omega_{[i]}(1),\cdots,\omega_{[i]}(L))\in\Omega_{[i]}^{L}$, if

$$\omega_{[i]}(l)\neq\omega_{[i]}(l')\quad\text{for all}\quad l\in\{1,\ldots,L\}\quad\text{and}\quad l'\in\{1,\ldots,L\}\setminus\{l\},$$

then

$$\sum_{l=1}^{L}E_{[i]}\Big[u_{[i]}(f(\omega_{[i]}(l),\omega_{-[i]}),\omega_{[i]}(l),\omega_{-[i]})\Big|\omega_{[i]}(l)\Big]$$
$$\geq\sum_{l=1}^{L}E_{[i]}\Big[u_{[i]}(f(\omega_{[i]}(l+1),\omega_{-[i]}),\omega_{[i]}(l),\omega_{-[i]})\Big|\omega_{[i]}(l)\Big],$$

where $E_{[i]}[\cdot|\omega_{[i]}]$ is the expectation operator over $\omega_{-[i]}$ conditional on $\omega_{[i]}$.

Condition 4 (Supermodularity): For every $i\in\{1,...,n\}$, $\Omega_{[i]}$ is an ordered set with $\geq$, and for every $\omega_{[i]},\omega'_{[i]},\omega''_{[i]},\omega'''_{[i]}\in\Omega_{[i]}$,

$$E_{[i]}\Big[u_{[i]}(f(\omega_{[i]},\omega_{-[i]}),\omega'_{[i]},\omega_{-[i]})\Big|\omega'_{[i]}\Big]+E_{[i]}\Big[u_{[i]}(f(\omega''_{[i]},\omega_{-[i]}),\omega'''_{[i]},\omega_{-[i]})\Big|\omega'''_{[i]}\Big]$$
$$\leq E_{[i]}\Big[u_{[i]}(f(\omega_{[i]}\vee\omega''_{[i]},\omega_{-[i]}),\omega'_{[i]}\vee\omega'''_{[i]},\omega_{-[i]})\Big|\omega'_{[i]}\vee\omega'''_{[i]}\Big]$$
$$+E_{[i]}\Big[u_{[i]}(f(\omega_{[i]}\wedge\omega''_{[i]},\omega_{-[i]}),\omega'_{[i]}\wedge\omega'''_{[i]},\omega_{-[i]})\Big|\omega'_{[i]}\wedge\omega'''_{[i]}\Big].$$

7.2. Results

As in the arguments in the previous sections, we can present the following theorem.

Theorem 5: *Suppose that the agents' private signals for each task are independent of each other, i.e.,*

$$p(\omega)=\prod_{i=1}^{n} p_{[i]}(\omega_{[i]}) \text{ for all } \omega\in\Omega .$$

Then, the following four properties hold.

Property 1: If there exists an infinite sequence of mechanisms $(\Gamma(K))_{K=1}^{\infty}$ *that virtually implement a social choice function* f *, Condition 3 holds.*

Property 2: Under Condition 3, a social choice function f *is virtually implemented by the infinite sequence of the linking mechanisms* $(\Gamma^{*}(K))_{K=1}^{\infty}$.

Property 3: Condition 4 implies Condition 3.

Property 4: Suppose that for every $i\in\{1,...,n\}$ *, there exists an increasing, continuous, and unbounded function* $e_i : R\to R$ *such that* $d_i^K(w_{[i]}^K)=\sum_{h=1}^{K} e_i(w_{[i],h}^K)$ *for all* $w_{[i]}^K\in R^K$ *. Then, a social choice function* f *is exactly implementable with respect to* K *if and only if Condition 3 holds.*

Properties 3 and 4 are easy to prove because we can directly apply the proofs of Propositions 3 and 4. However, we need to provide some explanations in order to prove Properties 1 and 2. The definition of virtual implementation in this section is different from that in the single agent case presented in Section 3. This is because we do not require the agents to play their best responses in the exact sense. However, even if we replace the original definition in Section 3 with that provided in this section, we can prove the necessity of Condition 1 for virtual implementation in exactly the same manner as in the proof of Theorem 1. Based on this, we can extend the necessity result obtained in the single agent case to the multiple agent case by simply applying the same logic as that used in the former case.

We need to provide more detailed explanations in order to prove Property 2. Let us assume any positive real number $\eta>0$ sufficiently close to zero and consider a sufficiently large K. Let $\Sigma_{[i]}(\eta,K)\subset\Sigma_{[i]}$ denote the set of agent i's strategies $\sigma_{[i]}$ such that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is less than η, i.e.,

$$E\left[\frac{\sharp\{h\in\{1,\cdots,K\}\mid m_{[i],h}\neq\omega_{[i],h}\}}{K}\middle|\Gamma^{*}(K),\sigma_{[i]}\right]<\eta,$$

where $E[\cdot|\Gamma^{*}(K),\sigma_{[i]}]$ is the expectation operator over ω conditional on $(\Gamma^{*}(K),\sigma_{[i]})$. As in Theorem 2, we can observe that $\Sigma_{[i]}(\eta,K)\subset\Sigma_{[i]}$ is nonempty for a sufficiently large K. Let us define

$$\varepsilon_{[i]}(\eta,K)\equiv\max_{\sigma_{-[i]}\in\Sigma_{-[i]}(\eta,K)}\Bigg[\max_{\sigma_{[i]}\in\Sigma_{[i]}}E\left[v^{K}_{[i]}(g^{K}(m),t^{K}_{[i]}(m),\omega^{K})\middle|\Gamma^{*}(K),\sigma\right]$$
$$-\max_{\sigma_{[i]}\in\Sigma_{[i]}(\eta,K)}E\left[v^{K}_{[i]}(g^{K}(m),t^{K}_{[i]}(m),\omega^{K})\middle|\Gamma^{*}(K),\sigma\right]\Bigg].$$

Note that there exists a $\max_{i\in\{1,\dots,n\}}\varepsilon_{[i]}(\eta,K)-$Nash equilibrium in $\Gamma^{*}(K)$. As in the proof of Theorem 2, we can observe that there exists a best response for agent i such that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is close to zero.[11] This implies that we can choose a strategy for agent i in $\Sigma_{[i]}(\eta,K)$ that is nearly a best response and, therefore, we can choose $\varepsilon_{[i]}(\eta,K)$ close to zero. In fact, by choosing η as close to zero and then choosing a sufficiently large K, we can obtain $\varepsilon_{[i]}(\eta,K)$ as close to zero as possible. Thus, we have proved Property 2.

7.3. Remarks

From Property 4, it follows as in Appendix B that we can replace Condition 3 with the condition that for every $i\in\{1,\dots,n\}$, there exists $r_{[i]}:\Omega_{[i]}\to R$ such that for every $\omega_{[i]}\in\Omega_{[i]}$ and every $\tilde{\omega}_{[i]}\in\Omega_{[i]}$,

[11] This does not imply that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is less than η. This is why we cannot use the exact Nash equilibrium in place of $\varepsilon-$Nash equilibrium in the case of multiple agents.

$$E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right]+r_{[i]}(\omega_{[i]})$$

$$\geq E_{[i]}\left[u_{[i]}(f(\tilde{\omega}_{[i]},\omega_{-[i]}),\omega)\middle|\omega_{[i]}\right]+r_{[i]}(\tilde{\omega}_{[i]}).$$

By assuming $r_{[i]}(\omega_{[i]})\equiv 0$ for all $i\in\{1,...,n\}$, we can verify that for Condition 3, it is sufficient that for every $i\in\{1,...,n\}$, every $\omega_{[i]}\in\Omega_{[i]}$, and every $\tilde{\omega}_{[i]}\in\Omega_{[i]}$,

$$E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right]\geq E_{[i]}\left[u_{[i]}(f(\tilde{\omega}_{[i]},\omega_{-[i]}),\omega)\middle|\omega_{[i]}\right].$$

This condition is the same as the ex ante efficiency that was introduced by Jackson and Sonnenschein (2005) as the sufficient condition for implementation. In contrast with the present paper, Jackson and Sonnenschein assumed private values and showed full implementation in that for a sufficiently large K, *every* Nash equilibrium in the linking mechanism virtually induces the value of the ex ante efficient social choice function.

Even from a practical viewpoint, linking mechanisms are more effective than incentive wage schemes. In fact, unlike incentive wage schemes, the linking mechanisms are not dependent on the details of the agents' payoff functions.[12]

In Theorem 5, we have supposed that the agents' private signals are independent of each other. Similarly, even in the case of *correlated* private signals across all the agents, we can prove that Properties 1, 2, and 3 hold and that the sufficient part of Property 4 holds. However, if the agents' signals are correlated to each other, the class of exactly implementable social choice functions is wider than the class of social choice functions that are virtually implementable by linking mechanisms. In fact, any social choice function is exactly implementable whenever the probability distribution of the other agents' signal profile conditional on each agent's private signal varies across her signals, i.e.,

$$p_{[i]}(\cdot\,|\,\omega_{[i]})\neq p_{[i]}(\cdot\,|\,\omega'_{[i]})\ \text{ for all }\ \omega_{[i]}\in\Omega_{[i]}\ \text{ and all }\ \omega'_{[i]}\in\Omega_{[i]}\setminus\{\omega_{[i]}\},$$

where $p_{[i]}(\omega_{-[i]}\,|\,\omega_{[i]})\equiv\dfrac{p(\omega)}{\sum_{\omega'_{-[i]}\in\Omega_{-[i]}}p(\omega'_{-[i]},\omega_{[i]})}$. Needless to say, this sufficient condition is extremely weak. See Crèmer and McLean (1985, 1988), Matsushima (1990, 2005),

[12] In the case of multiple agents, both linking mechanisms and incentive wage schemes depend on the probability function of each agent's signals. In the case of a single agent, however, incentive wage schemes need not be dependent on this distribution.

Aoyagi (1998), Chung (1999), and others. In this case, the incentive wage scheme for each agent depends on the other agents' announcements as well as on her own announcement. This implies that whether each agent should be punished or rewarded is crucially dependent on the *whistle-blowing* of the other agents. Thus, even though the linking mechanism is a potentially powerful tool to incentivize agents in the case of correlated private signals, the drawback of this mechanism as compared with incentive wage schemes is that whistle-blowing is never effective without side payments.

7.4. Macro Shock

Throughout this paper, we have assumed that the private signals were drawn independently across all the tasks. However, by just adding a prior message stage in a simple way, the linking mechanism does function effectively even if the private signals are *correlated across all the tasks*.

Consider a situation in which there exist three or more agents. Suppose that there exists a *macro* shock $\theta \in \Theta$ on which the probability distribution of ω_h for each task $h \in \{1,...,K\}$ and the social choice function are dependent. We denote

$$p(\omega) = p(\omega \mid \theta)\,,\quad p_{[i]}(\omega_{[i]}) = p_{[i]}(\omega_{[i]} \mid \theta)\text{, and}\quad f(\omega) = f(\omega,\theta)\,.$$

Here, we assume that Θ is a finite set, and for every $i \in \{1,...,n\}$, every $\theta \in \Theta$, and every $\theta' \in \Theta \setminus \{\theta\}$,

$$(17)\qquad p_{[i]}(\cdot \mid \theta) \neq p_{[i]}(\cdot \mid \theta')\,.$$

In order to be able to apply the appropriate linking mechanism, the principal needs to know the true macro shock θ. However, the principal and the agents both *cannot* observe this shock.

As we have already known, with a sufficiently large K, it is almost certain, based on the law of large numbers, that the realized proportion of the tasks for which an agent observes each private signal is almost the same as the probability of her observing this signal for a single task. This along with (17) implies that it is almost certain that each agent can infer the macro shock *correctly* from the observed private signals for all the tasks.

With three or more agents, the principal can incentivize the agents to tell of what they know about the macro shock to the best of their abilities as follows. The principal requires each agent to announce about the macro shock. If more than a half of the agents announce the same macro shock $\tilde{\theta} \in \Theta$, the principal will apply the linking mechanism associated with $p(\cdot) = p(\cdot \mid \tilde{\theta})$ and $f(\omega) = f(\omega, \tilde{\theta})$. If there is no such $\tilde{\theta}$, the principal will apply some fixed mechanism. Hence, announcing about the macro shock honestly is nearly a best response for each agent if the other agents announce honestly, because her announcement does not much influence which mechanism the principal will apply. This implies that truth-telling about the macro shock is described as an epsilon-Nash equilibrium strategy.

Unfortunately, this argument depends crucially on the assumption that there exist three or more agents. In fact, with a single agent, the principal needs to design an incentive wage scheme, in addition to the linking mechanism, in order to elicit the true macro shock from this single agent.[13]

[13] Currently we are preparing a paper on this subject.

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Appendix A

We will show that in the proof of Theorem 1, we can assume that for every K, $\Gamma(K)$ is symmetric. Suppose that $\Gamma(K)$ is not symmetric. For each permutation π on $\{1,...,K\}$, we define $g^{K,\pi}$ and $t^{K,\pi}$ by

$$g^{K,\pi}(m^\pi)[a^{K,\pi}] = g^K(m)[a^K] \quad \text{for all} \quad m \in M \quad \text{and all} \quad a^K \in A^K$$

and

$$t^{K,\pi}_{\pi(h)}(m^\pi) = t^K_h(m) \quad \text{for all} \quad m \in M \quad \text{and all} \quad h \in \{1,\ldots,K\}.$$

Let $\Gamma(K)^\pi = (M, g^{K,\pi}, t^{K,\pi})$. Note that for every $\sigma \in \Sigma$ and every $\omega^K \in \Omega^K$,

$$E\left[\sum_{h=1}^{K}\left\{u(a_h,\omega_h)+d(w_h)\right\}\middle|\Gamma(K),\sigma,\omega^K\right]$$
$$= E\left[\sum_{h=1}^{K}\left\{u(a_h,\omega_h^{K,\pi})+d(w_h)\right\}\middle|\Gamma(k)^\pi,\sigma^\pi,\omega^{K,\pi}\right],$$

where $E\left[\cdot\middle|\Gamma(K),\sigma,\omega^K\right]$ is the expectation operator over ω conditional on $(\Gamma(K),\sigma,\omega^K)$, $\omega^{K,\pi} = (\omega_1^{K,\pi},...,\omega_K^{K,\pi}) \in \Omega^K$ is defined by $\omega^{K,\pi}_{\pi(h)} = \omega_h$ for all $h \in \{1,\ldots,K\}$, and $\sigma^{K,\pi} = \sigma^\pi \in \Sigma$ is defined by $\sigma^\pi(\omega^{K,\pi}) = m^\pi$, where $m = \sigma(\omega^K)$, for all $\omega^K \in \Omega^K$. Since the truthful strategy $\hat{\sigma}$ is a best response in $\Gamma(K)$ and $\hat{\sigma}^\pi$ is also truthful, i.e., $\hat{\sigma}^\pi = \hat{\sigma}$, it follows from (4) that $\hat{\sigma}$ is a best response in $\Gamma(K)^\pi$, and

$$\lim_{K\to\infty} E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid a_h = f(\omega_h)\}}{K}\middle|\Gamma^\pi(K),\hat{\sigma}^{K,\pi}\right] = 1.$$

We define $\overline{\Gamma}(K) = (M, \overline{g}^K, \overline{t}^K)$ by

$$\overline{g}^K \equiv \sum_{\pi\in\Pi}\frac{1}{K!}g^{K,\pi} \quad \text{and} \quad \overline{t}^K \equiv \sum_{\pi\in\Pi}\frac{1}{K!}t^{K,\pi},$$

where Π denotes the set of permutations on $\{1,...,K\}$. It is clear that $\overline{\Gamma}(K)$ is symmetric, $\hat{\sigma}$ is a best response in $\overline{\Gamma}(K)$, and

$$\lim_{K\to\infty} E\left[\frac{\sharp\{h \in \{1,\cdots,K\} \mid a_h = f(\omega_h)\}}{K}\middle|\overline{\Gamma}(K),\hat{\sigma}^{K}\right] = 1.$$

Therefore, we can assume that for every K, $\Gamma(K)$ is symmetric.

Appendix B: Proof of Proposition 4

Let us select K arbitrarily. Suppose that a social choice function f is exactly implementable with respect to K, i.e., there exist a mechanism $\Gamma(K)$ and a best response $\sigma^* \in \Sigma$ in $\Gamma(K)$ such that

(B-1) $$g^K(\sigma^*(\omega^K))[f^K(\omega^K)] = 1 \text{ for all } \omega^K \in \Omega^K.$$

Consider any K' such that $K' = yK + z$ for some positive integer y and some integer $z \in \{0,...,K-1\}$. It is clear that by using $\Gamma(K)$ in a set of y for the first yK tasks, we can construct a mechanism $\Gamma(K')$ such that there exists a best response in $\Gamma(K')$ that induces the value of the social choice function f for the first yK tasks. This implies that there exists an infinite sequence of mechanisms that virtually implements f with respect to p. This along with Theorem 1 implies that Condition 1 is necessary for exact implementation.

Next, we will prove the sufficiency. We merely need to show that Condition 1 is sufficient in the case of $K = 1$ because if this is true, we can exactly implement the social choice function irrespective of K by simply using $\Gamma(K)$ in a set of K for all tasks. Thus, it is sufficient to verify whether or not there exists a side payment function $r : \Omega \to R$ such that

$$u(f(\omega),\omega) + r(\omega) \geq u(f(\tilde{\omega}),\omega) + r(\tilde{\omega}) \text{ for all } \omega \in \Omega \text{ and all } \tilde{\omega} \in \Omega.$$[14]

Using Theorem 1 proposed by Fan (1956) as it is used in D'Aspremont and Gèrard-Varet (1979, Theorem 7), we can show that a necessary and sufficient condition for the existence of such an r is that for every $\mu : \Omega^2 \to R_+ \bigcup \{0\}$, if

(B-2) $$\sum_{\tilde{\omega} \neq \omega \in \Omega} \{\mu(\omega,\tilde{\omega}) - \mu(\tilde{\omega},\omega)\} = 0 \quad \text{for all } \omega \in \Omega,$$

then

(B-3) $$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \neq \omega \in \Omega} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\} \mu(\omega,\tilde{\omega}) \geq 0.$$

For every $L \in \{2,\ldots,I\}$, an L-tuple of private signals $(\omega(1),\ldots,\omega(L)) \in \Omega^L$ is said to be a *cycle* if for every $l \in \{1,\ldots,L\}$,

$$\mu(\omega(l),\omega(l+1)) > 0$$

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[14] According to the revelation principle, without loss of generality, we can limit our attention to the direct mechanisms that have a side payment function $t : \Omega \to R$ such that for every $\omega \in \Omega$ and every $\tilde{\omega} \in \Omega$,

$$u(f(\omega),\omega) + e(t(\omega)) \geq u(f(\tilde{\omega}),\omega) + e(t(\tilde{\omega})).$$

This along with the unboundedness of e implies that there exists such an r in which $r \equiv e \circ t$.

and

$$\omega(l) \neq \omega(l') \text{ for all } l' \in \{1,\ldots,L\} \setminus \{l\}.$$

Suppose that Condition 1 holds, i.e., for every $L \in \{2,\cdots,I\}$ and every $(\omega(1),\cdots,\omega(L))$, if

$$\omega(l) \neq \omega(l') \in \Omega \text{ for all } l \in \{1,\ldots,L\} \text{ and all } l' \in \{1,\ldots,L\} \setminus \{l\},$$

then

(B-4) $$\sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) \geq \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l)).$$

It is evident that we can choose $\mu = \mu(1):\Omega^2 \to R_+ \cup \{0\}$, $\omega(1) \in \Omega$, and $\omega(2) \in \Omega \setminus \{\omega(1)\}$ satisfying (B-2) and

$$\mu(\omega(1),\omega(2)) > 0.$$

If $\mu(\omega(2),\omega(1)) > 0$ holds, then $(\omega(1),\omega(2))$ is a cycle. If $\mu(\omega(2),\omega(1)) = 0$, it follows from (B-2) that we can choose a private signal $\omega(3) \in \Omega \setminus \{\omega(1),\omega(2)\}$ such that

$$\mu(\omega(2),\omega(3)) > 0.$$

Let us select a positive integer l arbitrarily. Suppose that $(\omega(1),...,\omega(l-1))$ satisfies

$$\omega(l') \neq \omega(l'') \text{ for all } l' \in \{1,\ldots,l-1\} \text{ and } l'' \in \{1,\ldots,l-1\} \setminus \{l'\},$$

$$\mu(\omega(l'),\omega(l'+1)) > 0 \text{ for all } l' \in \{1,\ldots,l-2\},$$

and

$$\mu(\omega(l'),\omega(l'')) = 0 \text{ for all } l' \in \{2,\ldots,l-2\} \text{ and } l'' \in \{1,\ldots,l'-1\}.$$

If there exists $l' \in \{1,\ldots,l-2\}$ such that $\mu(\omega(l-1),\omega(l')) > 0$, then $(\omega(l'),...,\omega(l-1))$ is a cycle. If there exists no such l', it follows from (B-2) that we can choose a private signal $\omega(l) \in \Omega \setminus \{\omega(1),...,\omega(l-1)\}$ such that

$$\mu(\omega(l-1),\omega(l)) > 0.$$

Since $\sharp\Omega = I$ is finite, by continuing the above step, we can determine $l \in \{2,...,I\}$ and $l' \in \{1,....,l-1\}$ such that $(\omega(l'),...,\omega(l))$ is a cycle. By replacing l' and l with 1 and L, respectively, we denote this cycle by

$$C(1) \equiv (\omega(1),\ldots,\omega(L)).$$

Let

$$\xi(1) \equiv \min_{l \in \{1,\ldots,L\}} \mu(\omega(l),\omega(l+1)).$$

We specify $\eta(1):\Omega^2 \to R$ such that

$$\eta(1)(\omega(l),\omega(l+1)) = \xi(1) \text{ for all } l \in \{1,...,L\},$$

and for every $(\omega,\omega') \in \Omega^2$, if there exists no $l \in \{1,...,L\}$ such that

$$(\omega, \omega') = (\omega(l), \omega(l+1)),$$

then

$$\eta(1)(\omega, \omega') = 0.$$

From (B-4), it follows that

$$\sum_{(\omega,\omega')\in\Omega^2} \{u(f(\omega),\omega) - u(f(\omega'),\omega)\}\eta(1)(\omega,\omega') \geq 0.$$

We define $\mu(2): \Omega^2 \to R$ by

$$\mu(2) \equiv \mu(1) - \eta(1).$$

From (B-2) and the definition of $\eta(1)$, it follows that

$$\mu(2)(\omega, \omega') \geq 0 \quad \text{for all} \quad \omega \in \Omega \quad \text{and all} \quad \omega' \in \Omega$$

and

$$\sum_{\tilde{\omega}\neq\omega\in\Omega} \{\mu(2)(\omega,\tilde{\omega}) - \mu(2)(\tilde{\omega},\omega)\} = 0 \quad \text{for all} \quad \omega \in \Omega.$$

If $\mu(2)(\omega,\tilde{\omega}) = 0$ for all $\omega \in \Omega$ and all $\tilde{\omega} \in \Omega \setminus \{\omega\}$, the inequality (B-3) holds for $\mu = \mu(2)$, i.e.,

$$\sum_{\omega\in\Omega} \sum_{\tilde{\omega}\neq\omega\in\Omega} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\}\mu(2)(\omega,\tilde{\omega}) \geq 0.$$

If $\mu(2)(\omega,\tilde{\omega}) > 0$ for some $\omega \in \Omega$ and some $\tilde{\omega} \in \Omega \setminus \{\omega\}$, we can construct a cycle $C(2)$ and $\mu(3)$ as we did in $C(1)$ and $\mu(2)$.

By continuing the above step, we can determine a positive integer q, $\eta(q'): \Omega^2 \to R_+ \bigcup \{0\}$ for each $q' \in \{1,...,q-1\}$, and $\mu(q): \Omega^2 \to R_+ \bigcup \{0\}$ such that

$$\sum_{\omega\in\Omega} \sum_{\tilde{\omega}\in\Omega\setminus\{\omega\}} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\}\mu(q)(\omega,\tilde{\omega}) \geq 0,$$

for every $q' \in \{1,...,q-1\}$,

$$\sum_{(\omega,\omega')\in\Omega^2} \{u(f(\omega),\omega) - u(f(\omega'),\omega)\}\eta(q')(\omega,\omega') \geq 0,$$

and

$$\mu = \sum_{q'=1}^{q-1} \eta(q') + \mu(q).$$

These imply (B-3). Thus, we have proved Proposition 4.