Self-organizing Urban Hierarchy*

Takatoshi Tabuchi† and Jacques-François Thisse‡

12 April 2006

1 Introduction

Casual observation shows that cities differ by their size as well as by the various types of economic activities which they carry on. In this respect, what is probably the most striking feature of the space-economy is that cities form a hierarchical system exhibiting some regularity in terms of both their size and the array of goods they supply (Henderson, 1988). Another empirical regularity is that cities having the same size are more or less evenly distributed across places (Marshall, 1989). The purpose of central place theory, a major research topic in classical economic geography, is then to explain why such a regular urban system exists. According to Christaller (1933), Lösch (1940) and their successors, different markets are arranged in a way such that a city in which a good is supplied also provides the goods made available in a larger number of urban centers. Hence, all goods supplied in a central place of order $k = 1, 2, ...$ are also supplied in central places of higher order ($l > k$). Furthermore, the same authors also argue that, in a featureless space, cities of equal sizes are equally spaced. Accordingly, the urban system would be formed by a family of nested and regular lattices, with one lattice per good, each vertex of a lattice accommodating a city that supplies consumers situated in its vicinity. The *urban hierarchical principle* then holds when the number of goods supplied in a city rises with its size, while the spacing of cities having the same size is equal over a flat space.

The bulk of the research on central place theory has been directed towards identifying geometric conditions under which a superposition of regular structures is possible (the more

*We wish to thank Masahisa Fujita, Tomoya Mori, Se-il Mun, Yasuhiro Sato, Dao-Zhi Zeng and participants at ARSC Meetings 2005 and Kyoto University for helpful comments and discussions.

†Faculty of Economics, University of Tokyo (Japan)

‡CORE, Université catholique de Louvain (Belgium), PSE, Ecole nationale des ponts et chaussées (France), and CEPR

complete and elaborate study of this kind can be found in Alao *et al.*, 1977). These considerations are only interesting if they are based on microeconomic foundations. If there are no economic forces which lead firms of different types to cluster, it is hard to see why the central place system would be more likely to emerge than any other configuration. In this respect, the only economic contributions we are aware of are due to Eaton and Lipsey (1982), who develop a spatial competition model of central places, and to Quinzii and Thisse (1990), who retain the same approach to analyze its normative aspects and characterize the socially optimal configuration of central places. Unfortunately, those papers use a partial equilibrium setting in which consumers' locations are exogenous.

Henderson (1974, 1988) has developed a compelling and original approach that allows him to describe how a hierarchy of cities emerges. In each city, there is again a tension between two forces. On the one hand, there are external economies associated with the agglomeration of firms within a city. On the other hand, there are diseconomies generated by the need to commute in a more or less large city. Hence, in equilibrium, each city has a well-defined size that depends on the type of firms it accommodates. As cities vary in their industrial mix, they have different sizes because industries differ in the external economies they are able to create. However, in this approach, cities are like floating islands because nothing is said about city locations.

Fujita, Krugman and Mori (1999) take a first step in this direction by introducing several industrial and differentiated goods in an NEG-like model. As the population increases, they show that a more or less regular hierarchical central place system emerges within the economy. In this urban system, higher-order cities provide a larger number of groups of goods. In addition, there is two-way trade between cities because these cities supply differentiated goods. This leads to a more intricate pattern of trade in which horizontal relations are superimposed onto the pyramidal structure of central place theory. As expected, higher-order cities export a larger variety of goods than lower-order cities. However, horizontal relations between cities of the same order may be more important than trade with lower-order cities. The urban hierarchies that emerge can be more complex than in the simple Christaller model of central places, and indeed, often reflect the urban systems that actually appear in modern economies. However, their approach is numerical, whereas our model is analytically-solvable. Another distinguishing feature is that our economic geography is of the putty-clay type: once they exist, *cities are sticky.*

The purpose of this paper is precisely to show that the urban hierarchical principle may be derived in an NEG-type model, in which the number, size, and location of cities are determined endogenously. Our main purpose is to focus on the size and the location of

cities (the *urban* aspect) as well as the spatial distribution of each industry across cities (the *industrial* aspect) when consumers/workers are free to choose where to live and for which industry to work. To achieve our objective, we build a general equilibrium model with monopolistically competitive markets for the industrial sectors in which the locations of firms are interdependent. Consumers are attracted by places as they are in multipurpose shopping models but are allowed to choose their location where they compete on the local labor market. As in Fujita, Krugman and Mori, we retain the basic ingredients of NEG to generate the agglomeration of firms and consumers. In particular, cities are service suppliers to all the agricultural regions as well as to the other urban centers, the reason being that consumers have a preference for variety whereas firms produce differentiated goods.

Before proceeding, it is worth noting that a difficulty seems to be inherent to central place theory. It is related to the integer problem encountered in club theory (Scotchmer, 2002), but exhibits some new and additional features. Indeed, for different lattices to be nested, it must be that the number of cities of order k is just twice the number of cities of order $k-1$ and half the number of cities of order $k+1$. This explains why a central place configuration can emerge only for power values of 2.

2 The model

The economy involves agricultural and industrial activities. In the agricultural sector, a homogenous good is produced under constant returns, perfect competition and zero transport cost; this good is taken as the numéraire. There are $I(\geq 1)$ industrial sectors, each producing a differentiated good under increasing returns, monopolistic competition and positive transport costs. The economy has $C(\geq 1)$ cities - or central places - $c = 1, 2, \ldots, C$ located at $x_c \in (0, 1]$ on a circumference of length 1. The location and size of cities are endogenous. Throughout the paper, cities are described by subscripts and industries by superscripts.

There are two production factors, farmers and workers. Each farmer and worker is endowed with one unit of labor. The exogenously given masses of farmers and workers are given by $1 - \mu > 0$ and $\mu > 0$, respectively. Farmers are immobile and uniformly distributed along the circumference with a density equal to $1 - \mu$. Workers can migrate freely across cities $c = 1, 2, \ldots, C$ and industries $i = 1, 2, \ldots, I$. Let λ_c^i be the endogenous number of workers in city c and industry i with $\sum_{i=1}^{I} \sum_{c=1}^{C} \lambda_c^i = \mu$. Whereas the number of industries I is exogenous, the number of cities C as well as the distribution of workers across cities and industries λ_c^i are endogenous.

The technologies are as follows. In order to produce one unit of the homogenous good, one

farmer is needed. As the technology exhibits constant returns, the equilibrium wage equals 1 in the agricultural sector. To produce one unit of a variety of the differentiated good i, a firm needs a fixed requirement of $f^i > 0$ workers and a marginal requirement of $m^i > 0$ units of the numéraire. Hence, it must be that

$$f^i n_c^i = \lambda_c^i \qquad \text{for all } c \text{ and } i \tag{1}$$

Because we allow for both spatial and sectoral mobility, there is no obvious and natural way to model the choice process of workers. As workers show very little spatial mobility in most industrialized countries, we assume that they, first, choose where to live and, then, which job to take. Given this assumption, our setting involves three stages. In the first one, a worker chooses a city where to live; in the second stage, she selects an industry in which to work; in the third stage, each firm chooses prices and each consumer (= worker or farmer) chooses her consumption of the differentiated goods, made available in each city c by industry i, as well as her consumption of the homogenous good. As usual, we solve the subgame-perfect Nash equilibria by backward induction.

(i) In the last stage, firms select prices so as to maximize profits conditional upon consumers' demands. The distribution of workers, and then the distribution of firms, across cities and sectors being given, consumers maximize their utility given by

$$U = \sum_{i=1}^{I} \alpha^i \log Q^i + H \tag{2}$$

where

$$Q^i \equiv \left[\int_0^{n^i} q^i(v)^{\frac{\sigma i - 1}{\sigma i}} \mathrm{d}v \right]^{\frac{\sigma i}{\sigma i - 1}}$$

with $\alpha^i > 0$.[1] In this expression, H is the consumption of the homogenous good, Q^i the composite good associated with industry i, $q_c^i(v)$ the consumption of variety v produced by a firm belonging to industry i and located in city c, whereas n^i is the mass of varieties supplied by industry i. The parameter $\sigma^i > 1$ measures both the own- and cross-price elasticities of demand for any variety of good i.

The budget constraint of a consumer living at $x \in (0, 1]$ is as follows:

$$\sum_{c=1}^{C} \sum_{i=1}^{I} \int_0^{n_c^i} p_c^i(v; x) q_c^i(v) \, \mathrm{d}v + H = w(x)$$

[1] This is the multi-industry extension of the utility used by Martin and Rogers (1995) and Pflüger (2004). It slightly differs from Krugman's (1991) utility in that the homogenous good H is multiplicative in Krugman, whereas it is additive here. This vastly simplifies the overall analysis.

where $p_c^i(v;x)$ is the delivered price at x of variety v of good i produced in city c, and $w(x)$ the income she earns at location x. By symmetry, all i-firms set up in the same city choose the same price so that we may drop the variable v in what follows.

The maximization of utility (2) yields the following individual demand in location x for a variety of good i produced in location x_c:

$$q_c^i(x) = \alpha^i \frac{p_c^i(x)^{-\sigma^i}}{P^i(x)^{1-\sigma^i}} \tag{3}$$

where $p_c^i(x)$ is the common delivered price of a variety of good i produced in city c and consumed at x and where

$$P^i(x) = \left[\sum_{c=1}^{C} n_c^i p_c^i(x)^{1-\sigma^i}\right]^{\frac{1}{1-\sigma^i}} \tag{4}$$

is the price index of good i that prevails at x. Accordingly, the indirect utility of a consumer working for industry i in city c is as follows:

$$V_c^i = \sum_{j=1}^{I} \alpha^j \left(\log \alpha^j - 1 - \log P_c^j\right) + w_c$$

where P_c^j is the price index of good j in city c and w_c the income of a consumer residing in city c. Because of the intersectoral mobility of workers, wages are the same across industries within each city but they may vary across cities.

Let $|x_c - x|$ be the shortest distance between city c and location x along the circumference. Then, $\tau_c^i(x) \equiv m^i \tau^{|x_c - x|} > 1$ is the number of units of the numéraire that a firm producing good i and located in city c has to bear to produce and ship one unit of its output to location x. Without loss of generality, we choose the unit of good i for m^i to be equal to 1 In this case, the parameter τ, which we call the *transport rate*, is a measure of the impediments to trade any differentiated good across locations. It is assumed to be the same for all differentiated goods.

The profit of a firm belonging to industry i and located in city c is then given by

$$\begin{aligned} \pi_c^i &= \sum_{d=1}^{C} \left[p_c^i(x_d) - \tau_c(x_d)\right] q_c^i(x_d) \sum_{j=1}^{I} \lambda_d^j \\ &\quad + (1-\mu) \int_0^1 \left[p_c^i(x) - \tau_c(x)\right] q_c^i(x)\, \mathrm{d}x - f^i w_c \end{aligned} \tag{5}$$

where λ_d^j is the number of consumers working for industry j in city d. The maximization of (5) yields

$$p_c^i(x) = \frac{\sigma^i}{\sigma^i - 1} \tau_c(x) \qquad \text{for all } c \text{ and } i \tag{6}$$

Hence, the equilibrium price index (4) becomes

$$P^i(x) = \left[\frac{\sigma^i}{\sigma^i - 1}\sum_{c=1}^{C} n_c^i \phi_c^i(x)\right]^{\frac{1}{1-\sigma^i}} \qquad \text{for all } i \tag{7}$$

where $\phi_c^i(x) \equiv \tau_c(x)^{1-\sigma^i}$ is to be interpreted as a measure of the accessibility of a consumer at x to any variety of good i produced in city c. Hence, the sum in (7) may be interpreted as the *market accessibility* of a consumer at x to the whole array of varieties provided in all cities. This sum rises as more firms are set up close to the consumer, which in turn leads to more competition and, therefore, to a lower price index through lower transport costs.

Workers' wage is determined through a bidding process in which firms belonging to industry i compete to hire them. As a result, the wage bill of a firm is less than ($\lambda_c^i = 0$) or equal to its gross profits ($\lambda_c^i \geq 0$). These I equations together with the total number of workers who have chosen to live in c allows for the determination of the shares $\lambda_c^i \geq 0$ and of the wage $w_c \geq 0$.

As workers have already chosen where to live, the population

$$\Lambda_c \equiv \sum_{j=1}^{I} \lambda_c^j \qquad \text{for all } c$$

of each city c is given. Plugging (6) into (5) and using the zero-profit condition together with the labor market equilibrium condition (1), we get

$$w_c = \frac{\alpha^i}{\sigma^i}\left[\sum_{e=1}^{C}\frac{\Lambda_e \phi_c^i(x_e)}{\sum_{d=1}^{C}\phi_d^i(x_e)\lambda_d^i} + \int_0^1 \frac{(1-\mu)\phi_c^i(x)}{\sum_{d=1}^{C}\phi_d^i(x)\lambda_d^i}\mathrm{d}x\right] \qquad \text{for all } i$$

Then, the indirect utility of a consumer working in industry i and city c becomes:

$$\begin{aligned} V_c^i \;=\; & \sum_{j=1}^{I}\sum_{d=1}^{C}\frac{\alpha^j}{\sigma^j - 1}\log\left(\phi_c^j(x_d)\lambda_d^j\right) \\ & + \frac{\alpha^i}{\sigma^i}\left[\sum_{e=1}^{C}\frac{\Lambda_e \phi_c^i(x_e)}{\sum_{d=1}^{C}\phi_d^i(x_e)\lambda_d^i} + \int_0^1 \frac{(1-\mu)\phi_c^i(x)}{\sum_{d=1}^{C}\phi_d^i(x)\lambda_d^i}\mathrm{d}x\right] \end{aligned} \tag{8}$$

The first term of (8) plays the role of an agglomeration force because it expresses the access of a worker living in c to the whole array of goods and varieties. By contrast, the second and third terms have the nature of a dispersion force. Indeed, a more concentrated pattern of firms, which allows for a better access, leads to more competition among firms, thus yielding lower gross profits, whence a lower wage for that worker.

(ii) In the second stage, a worker has to choose the industry that gives her the highest utility level, anticipating her consumption of all goods. A *sectoral equilibrium* in city c is such that

$$V_c^i \leq \overline{V}_c \qquad \text{and} \qquad \left(V_c^i - \overline{V}_c\right)\lambda_c^i = 0 \qquad \text{for all } i \tag{9}$$

Thus, when $\lambda_c^i > 0$, it must be that all workers living in c reach the same utility level. The existence and uniqueness of such an equilibrium are analyzed in section **??**.

(iii) In the first stage, each worker selects the city that gives her the highest utility level in each industry, anticipating the industry for which she will work as well as her consumption. A *spatial equilibrium* for industry i's workers is then such that

$$V^i(x) \leq \overline{V}^i \qquad \text{and} \qquad [V^i(x) - \overline{V}^i]\lambda_c^i = 0 \qquad \text{for all } x \in (0,1] \tag{10}$$

because workers are free to establish a new city founded at x. As usual in NEG models, the stability analysis is conducted by means of the replicator dynamics of the C variables for each industry i:

$$\dot{\lambda}_c^i = \lambda_c^i \left(V_c^i - \sum_{d=1}^{C} \frac{\lambda_d^i}{\Lambda^i} V_d^i \right) \equiv J_c \qquad \text{for all } c \tag{11}$$

where $\cdot$ denotes the time derivative of λ_c^i. The existence and stability of equilibrium are analyzed in section **??**.

3 Symmetric and asymmetric equilibria

In this section, because of the integer problem discussed in the introduction, we focus on equilibria involving nested patterns of cities in the case where the number C of cities is a power of 2.

Given the outcome of the third stage, we show below that the second stage has a single equilibrium.

Proposition 1 *In each city, there exists a unique sectoral equilibrium.*

The proof is given in Appendix A.

Since the equilibrium outcome of the second and third stages is unique, we may move directly to the first stage. As in most NEG models, this equilibrium need not be symmetric, unique or stable, thus making the analysis much more involved than in the other two stages.

3.1 The city-doubling point

A symmetric equilibrium with $C = 2^k$ cities, k being an integer larger than or equal to 1, implies that all cities are located equidistantly and have the same population size. It remains to show that the share of each industrial sector is the same across cities. By solving the sectoral equilibrium condition $V^i = \overline{V}$ for all i, the equilibrium share is uniquely determined by

$$\lambda_c^i = \lambda^i \equiv a^i \frac{\mu}{C} \qquad \text{for all } c \text{ and } i \tag{12}$$

where

$$a^i \equiv \frac{\frac{\alpha^i}{\sigma^i}}{\sum_j \frac{\alpha^j}{\sigma^j}}$$

Thus, the share of industry i depends on the elasticity of substitution (σ^i) and the salience coefficient of its output in individual preferences (α^i). Because cities are identical, we may delete the subscript c.

For (12) to be a symmetric spatial equilibrium with C cities, it must be that $V^i \geq V^i(x)$ holds for all i and all $x \in (0, 1]$.

It is shown in Appendix B that $V^i(x)$ has at most three local maxima arising at $x = 0$, $x = 1/2C$ and $x = 1/C$ over the interval $[0, 1/C]$, where $1/C$ is the distance between two adjacent cities in the symmetric configuration with C cities. Therefore, for (12) to be a spatial equilibrium, it must be that $V^i(0) \geq V^i(1/2C)$ holds for all i.

In order to work with tractable expressions, we rewrite the equilibrium condition $V^i(0) \geq V^i(1/2C)$ as follows:

$$g\left(\varphi^i\right) \geq 0 \qquad \text{for all } i$$

where

$$\begin{aligned} g\left(\varphi^i\right) \equiv\ & \left(\sigma^i - 1\right)\left(1 - \varphi^i\right)\left[\mu\left(1 + 3\left(\varphi^i\right)^2\right)\left(1 + \varphi^i\right) - \left(1 + 3\varphi^i\right)\left(1 + \left(\varphi^i\right)^2\right)\right]\log\varphi^i \\ & + \left[2a^i\mu\sigma^i\varphi^i - (1-\mu)\left(\sigma^i - 1\right)\left(1 - \left(\varphi^i\right)^2\right)\log\varphi^i\right]\left(1 + \left(\varphi^i\right)^2\right)\log\frac{2}{\varphi^i + \left(\varphi^i\right)^3} \\ & + 4a^i\mu\sigma^i\varphi^i\left(1 + \left(\varphi^i\right)^2\right)\left(\log\varphi^i\right)^2 \end{aligned}$$

and where $\varphi^i \equiv \left(\phi^i\right)^{\frac{1}{C}} = \tau^{\frac{1-\sigma^i}{C}} \in (0, 1)$ involves the transport rate τ and the elasticity of substitution σ^i, as in ϕ^i, but the number C of cities too.

It is readily verified that $\lim_{\varphi^i \to 0_+} g\left(\varphi^i\right) < 0$ and $g(1) = g'(1) = g''(1) = 0 > g'''(1)$. Hence, for any given value of C and for each i, the equation $g\left(\varphi^i\right) = 0$ has at least one positive solution. Let $\tau_2^i(C)$ be the largest of these solutions for each $i = 1, ..., I$, and $\tau_2(C) \equiv \min_i \tau_2^i(C)$, which we call the *city-doubling point* for a reason that will become clear below.

Thus, as long as $\tau < \tau_2(C)$, we have $V^i(0) > V^i(1/2C)$ so that there is no city at $x = 1/2C$, $x = 3/2C, \ldots$, $x = (2C-1)/2C$. Conversely, when $\tau > \tau_2$, the configuration with C cities is not an equilibrium because new cities emerge at $x = 1/2C$, $x = 3/2C, \ldots$, $x = (2C-1)/2C$.

Thus, we have:

Proposition 2 *Let C be any even number. Then, there exists a unique threshold $\tau_2(C) > 0$ such that the equilibrium number of cities at the symmetric configuration is equal to C when the transport rate is slightly below $\tau_2(C)$. Furthermore, the number of cities jumps to $2C$ when the transport rate is slightly above $\tau_2(C)$.*

Thus, if the initial number of cities is 2^k, the number of cities is successively divided by 2 as transport costs steadily decline. In other words, decreasing transport costs leads to a greater spatial concentration of the industrial sectors within a smaller number of central places. Eventually, when transport costs are sufficiently low, one city accommodates the whole industry. This explains why $\tau_2(C)$ is called a city-doubling point.

3.2 The symmetry-breaking point

Our next task is to study the conditions under which the symmetric outcome is stable. It is shown below that the symmetric equilibrium (12) with C cities is stable if all the real parts of the eigenvalues of the Jacobian matrix of (11) are negative. Using (18) given in Appendix C, this condition can be rewritten as follows:

$$h\left(\varphi^i\right) \geq 0 \qquad \text{for } i = 1, 2, ..., I \tag{13}$$

where

$$\begin{aligned} h\left(\varphi^i\right) \equiv & \left[\left(\sigma^i - 1\right)\left(1 + 2\left(1-\mu\right)\varphi^i + \left(\varphi^i\right)^2\right) - a^i\mu\left(2\sigma^i - 1\right)\left(1+\varphi^i\right)^2\right]\log\varphi^i \\ & +2\left(1-\mu\right)\left(\sigma^i - 1\right)\left(1 - \left(\varphi^i\right)^2\right) \end{aligned} \tag{14}$$

Because $\partial^2 h\left(\varphi^i\right)/\partial\tau^2 < 0$, h is strictly concave. Furthermore, as $\lim_{\tau\to 1} h(\tau) = 0$ and $\lim_{\tau\to\infty} h\left(\varphi^i\right) < 0$, it must be that, for each i, there exists a single positive solution $\tau = \tau_1^i$ to the equation $h\left(\varphi^i\right) = 0$. Let $\tau_1 \equiv \max_i \tau_1^i$, which we call the *symmetry-breaking point.*

The following result is then proven in Appendix C.

Proposition 3 *Let C be any even number. Then, there exists a unique threshold $\tau_1(C) > 0$ such that the symmetric configuration with C cities is stable (resp., unstable) when the transport rate is slightly below (resp., above) $\tau_1(C)$.*

From Propositions 2 and 3, we may conclude that there exists a symmetric stable equilibrium with C cities as long as τ belongs to $(\tau_1(C), \tau_2(C))$. However, for such an equilibrium to arise, it must be that $\tau_1(C) < \tau_2(C)$, for otherwise the symmetric configuration with C cities is never a stable equilibrium. Furthermore, as both τ_1 and τ_2 increase with C, it should be clear that the economy involves a decreasing number of cities when transport costs steadily decline.

3.3 When do city sizes differ?

In order to describe the evolution of the economy with respect to transport costs, we consider a steadily decrease in the parameter τ. When τ reaches the break point τ_1, the symmetric equilibrium ceases to be stable. Two cases may then arise: either the equilibrium involves a smaller number of cities or it retains the same number of cities but their size changes continuously.

In the former case, it appears to be very hard, if not impossible, to characterize the number and size of cities because we do not know how to select an equilibrium when there are multiple equilibria. By contrast, in the latter case, the original configuration can be shown to become an *alternating symmetric equilibrium*, which is defined as follows:

$$\lambda^i_{2c} = \lambda \quad \text{and} \quad \lambda^i_{2c-1} = 2\lambda^{i*} - \lambda \qquad \text{for } \lambda \in \left[0, 2\lambda^{i*}\right] \text{ and } c = 1, 2, \ldots, C/2 \tag{15}$$

where λ^{i*} is given by (12). At such a configuration, *cities with different sizes and different industrial mixes coexist*, one small city alternating with one large city.

This configuration (15) is a pitchfork bifurcation, and is stable, whereas the symmetric configuration is unstable. Following Rasband (1990), the conditions for such a bifurcation to occur at the symmetry-breaking point $\tau = \tau_1$ are as follows:

$$\begin{aligned}
&\mathrm{d}V^i\left(\lambda^{i*} - \lambda\right) = -\mathrm{d}V^i\left(\lambda^{i*} + \lambda\right) \\
&\left.\frac{\partial \mathrm{d}V^i(\lambda)}{\partial \lambda}\right|_{\lambda=\lambda^{i*},\ \tau=\tau_1} = 0 \\
&\left.\frac{\partial^2 \mathrm{d}V^i(\lambda)}{\partial \lambda \partial \tau}\right|_{\lambda=\lambda^{i*},\ \tau=\tau_1} > 0 \\
&\left.\frac{\partial^3 \mathrm{d}V^i(\lambda)}{\partial \lambda^3}\right|_{\lambda=\lambda^{i*},\ \tau=\tau_1} < 0
\end{aligned} \tag{16}$$

for all i. Alternating symmetry implies that the first three conditions are always met. However, whether or not the last condition (16) holds depends on the parameter values. In other

words, (16) appears to be the critical condition for the existence of an alternating symmetric equilibrium. This is the topic covered in the section below.

4 The urban hierarchy principle

4.1 The one-industry case

We first consider the case of one industry ($I = 1$) in which large and small cities may coexist. Hence, we may drop the superscript i. We know that the symmetric equilibrium bifurcates when the two conditions $g(\varphi) = 0$ and $\partial^3 \mathrm{d}V(\lambda)/\partial\lambda^3|_{\lambda=\mu/C} < 0$ hold. Therefore, we must construct the boundary of the domain for which the bifurcation is a pitchfork (see (ii) in Figure 1).

Consider the two equations $\partial^3 \mathrm{d}V(\lambda)/\partial\lambda^3|_{\lambda=\mu/C} = 0$ and $g(\varphi) = 0$ for which there is or there is no bifurcation. They have no explicit solutions, but we can define the implicit functions $\sigma = \sigma_{12}(\mu)$ and $\sigma = \sigma_1(\mu)$. Indeed, when $C = 2$ we have $\mu = \mu_{12}(\varphi)$ and $\sigma = \sigma_{12}(\varphi)$, whereas $\mu = \mu_1(\varphi)$ and $\sigma = \sigma_1(\varphi)$ when $C = 2^2, 2^3, \ldots$ As μ and σ are uniquely determined for all relevant values of φ (see Appendix D), we may define the implicit functions $\sigma = \sigma_{12}(\mu)$ and $\sigma = \sigma_1(\mu)$, which leads to the following.

Lemma 1 *A symmetric equilibrium bifurcates (resp., breaks) at $\tau = \tau_1$ if $\sigma < \sigma_{12}(\mu)$ (resp. $\sigma > \sigma_{12}(\mu)$) for $C = 2$, and if $\sigma < \sigma_1(\mu)$ (resp. $\sigma > \sigma_1(\mu)$) for $C = 2^2, 2^3, \ldots$*

We now come to the second section of the boundary and the two equations $g(\varphi) = 0$ and $h(\varphi) = 0$ for which the symmetric equilibrium exists or not. We first get $\mu = \mu_2(\varphi)$ and $\sigma = \sigma_2(\varphi)$. It can be shown that (**??**) with $a^i = 1$ is always satisfied whenever $\sigma > \sigma_2(\mu)$ holds. Hence, we have:

Lemma 2 *Assume $\tau_1 < \tau < \tau_2$. A stable symmetric equilibrium exists (resp., does not exist) if $\sigma > \sigma_2(\mu)$ (resp., $\sigma < \sigma_2(\mu)$).*

Based on these two lemmas together with the fact that $\sigma_2(\mu) < \sigma_{12}(\mu) < \sigma_1(\mu)$, we have proven the following. For a steadily decreasing transport cost τ, four cases may arise. (i) If $\sigma < \sigma_2(\mu)$, a stable symmetric equilibrium never exists. (ii) If $\sigma_2(\mu) < \sigma < \sigma_{12}(\mu)$, there exists a stable symmetric equilibrium for $\tau_1 < \tau < \tau_2$, which bifurcates at $\tau = \tau_1$. (iii) If $\sigma_{12}(\mu) < \sigma < \sigma_1(\mu)$, there exists a stable symmetric equilibrium for $\tau_1 < \tau < \tau_2$, which breaks at $\tau = \tau_1$ when $C = 2$ and bifurcates at $\tau = \tau_1$ when $C = 2^2, 2^3, \ldots$ (iv) If $\sigma > \sigma_2(\mu)$,

there exists a stable symmetric equilibrium for $\tau_1 < \tau < \tau_2$, which breaks at $\tau = \tau_1$. The four cases are depicted in Figure 1.

Case (ii) is especially relevant to us because it provides a foundation for the urban hierarchical principle.

Proposition 4 *Consider case (ii) in which the economy has $C = 2^{K+1}$ cities located equidistantly on the circumference, K being an integer. Then, for $k = K, K-1, \ldots, 1$, the evolutionary process associated with a steadily falling transport rate τ is given by the repetition of the following two types of equilibria: (a) 2^{k+1}-alternating symmetric equilibrium for $\tau_2\left(2^k\right) < \tau < \tau_1\left(2^{k+1}\right)$; (b) 2^k-symmetric equilibrium for $\tau_1\left(2^k\right) < \tau < \tau_2\left(2^k\right)$; (c) partially agglomerated equilibrium for $\tau_2(1) < \tau < \tau_1(2)$; and (d) fully agglomerated equilibrium for $0 < \tau < \tau_2(1)$.*

This pattern corresponds to the one investigated in central place theory à la Christaller-Lösch. What distinguishes our approach from the existing contributions is the fact that firms' and workers' locations are related here through a full-fledged micro-economic model.

Case (i) resembles to what is called the 'no-black-hole' condition in NEG: the manufacturing share μ is large and the elasticity of substitution σ low. However, there is no-black-hole condition here because, in our setting, the farming population is uniformly distributed along the circumference. It can readily be verified that full agglomeration (i.e. the black hole) is never a stable equilibrium for sufficiently large τ even though μ is large and close to 1 and/or σ is small and close to 1. The no-black-hole condition holds only if the population of farmers is atomistically distributed across cities. Note that the largest possible number of cities, as well as their locations, is predetermined in the case of an atomistic distribution of farmers in Krugman (1993), whereas it is endogenously determined in our case. Since case (i) does not satisfy the symmetric equilibrium condition, a stable equilibrium is necessarily asymmetric, involving some large and small cities. Unfortunately, such asymmetric equilibria are analytically intractable and do not agree with the urban hierarchical principle of equally-spaced, equal-size cities.

Case (iv) is the opposite to case (i) in that the manufacturing share μ is small and the elasticity of substitution σ is high, so that the symmetric equilibrium is likely to be stable. When the transport cost τ falls below the threshold τ_1, the symmetric equilibrium breaks and multiple equilibria may emerge. There is a tomahawk bifurcation, which also appears in Krugman (1991).[2] The multiplicity of equilibria prevents us to determine which stable

[2] When $\mu = \mu_2(\sigma)$ or $\mu = \mu_{22}(\sigma)$, the symmetric equilibrium also breaks but there is no multiplicity of equilibria (Ottaviano *et al.*, 2002).

equilibrium is selected after the symmetry breaks. For those reasons, we do not further investigate this case.

Finally, case (iii) lies between cases (ii) and (iv), we focus only on case (ii) and assume

$$\sigma_2(\mu) < \sigma < \sigma_{12}(\mu) \tag{17}$$

which is the shaded area in Figure 1. Denote τ_1 and τ_2 by $\tau_1(C)$ and $\tau_2(C)$ respectively when the number of cities is C. It can be readily shown that

$$\frac{\partial \tau_h(C)}{\partial \sigma} < 0 \qquad \frac{\partial \tau_h(C)}{\partial \mu} > 0 \qquad \text{for } h = 1, 2$$

for all relevant ranges of σ, μ and C. This confirms standard results in NEG for which symmetry is a stable outcome once τ is large, σ is large, or μ is small.

4.2 The multi-industry case

Consider the case of $I > 1$ industries. As in the foregoing section, for a symmetric equilibrium to exist, we assume

$$\sigma_1(\mu) < \sigma^i < \sigma_{12}(\mu)$$

When $C = 2^{K+1}$ cities are located equidistantly on the circumference, Proposition 2 may be easily extended to describe the impact of a falling transport cost τ on the way the urban system is organized.

The equilibrium path starts from iteration **0** and involves the repetition of **I+1** types of equilibria.

(**0**) 2^{K+1}-symmetric equilibrium for $\tau_1\left(2^K\right) < \tau < \tau_2\left(2^{K+1}\right)$, where all 2^{K+1} cities have all industries: *all industries are dispersed through the largest possible number of cities because transport costs are very large.*

For $k = K, K-1, \ldots, 1$,

(**1**) 2^{k+1}-alternating symmetric equilibrium for $\tau < \tau_1\left(2^{k+1}\right)$, where all 2^{k+1} cities have all industries;

(**2**) 2^{k+1}-alternating symmetric equilibrium with an agglomeration of order 1, where 2^k cities have all industries and whereas 2^k cities have $I-1$ industries;

(**3**) 2^{k+1}-alternating symmetric equilibrium with an agglomeration of order 2, where 2^k cities have all industries and whereas 2^k cities have $I-2$ industries;

$\vdots$

(**I**) 2^{k+1}-alternating symmetric equilibrium with an agglomeration of order $I-1$ for $\tau > \tau_2\left(2^k\right)$, where 2^k cities have all industries and another 2^k cities have 1 industry: *as the*

transport cost decreases, half of the cities accommodate a number of industries that steadily decreases to one;

(**I+1**) 2^k-symmetric equilibrium for $\tau_1\left(2^k\right) < \tau < \tau_2\left(2^k\right)$, where the number of cities is divided by two but all 2^k cities have all industries: *as the transport cost falls sufficiently, half the cities have disappeared.*

Repeating this process from (**1**) to (**I+1**) K times, we must end up with an urban configuration in which at most two cities are active:

(**IK+K+1**) for $\tau < \tau_1(2)$, the equilibrium involves one city having all industries as well as a second city has $I-1$ industries;

(**IK+K+2**) the equilibrium involves one city accommodating all industries whereas a second city has $I-2$ industries;

$\vdots$

(**IK+K+I**) the equilibrium has one city with all industries and a second city with a single industry;

(**IK+K+I+1**) for $0 < \tau < \tau_2(1)$, all industries are agglomerated within a single city: *as transport costs sufficiently fall, the number of cities steadily declines and full agglomeration arises in a single city as in standard two-region NEG models* .

In sum, we have identified four types of equilibria: alternating symmetry (**1**), alternating symmetry that obey the urban hierarchical principle (**2**)-(**I**) and (**IK+K+1**)-(**IK+2K**), full symmetry (**0**) and (**I+1**), and full agglomeration (**IK+2K+1**). In steps (**2**)-(**I**) and (**IK+K+1**)-(**IK+2K**), industries get concentrated within a smaller number of cities. More precisely, the number of industries decreases, first, in half of the cities and, then, the number of cities decreases by one half. Eventually, the number of cities fall and, when transport costs are sufficiently low, all industries are agglomerated in a single city, thus confirming one of the main results of NEG. Such equilibria with three industries are represented in Figure 3: full symmetry (I, IV and VII), the urban hierarchical principle (II, III, V, VI, VIII and XI) and full agglomeration (X). What makes our analysis new and original is the result that *central places are endogenous and arise when transport costs take intermediate values.* In addition, the transition from full dispersion to full agglomeration is progressive in our setting.

5 Conclusion

We have considered a general equilibrium model with monopolistically competitive markets, in which urban centers are service suppliers to all the agricultural regions as well as to the

other urban centers. We have retained the forward and backward linkages of NEG to generate the agglomeration of firms and workers in cities. Our main result is that central places arise endogenously when transport costs take intermediate values.

Appendices

A Proof of Proposition 1

Without loss of generality, we may assume that all industries are active in city c; otherwise we restrict ourselves to set of active industries.

Differentiation of V_c^i in (8) yields

$$\frac{\partial V_c^i}{\partial \lambda_c^i} = -\frac{\alpha^i}{\sigma^i}\left[\sum_{e=1}^{C} \frac{\Lambda_e\left(\phi_c^i(x_e)\right)^2}{\left(\sum_{d=1}^{C}\phi_d^i(x_e)\lambda_d^i\right)^2} + \int_0^1 \frac{(1-\mu)\left(\phi_c^i(x)\right)^2}{\left(\sum_{d=1}^{C}\phi_d^i(x)\lambda_d^i\right)^2}\mathrm{d}x\right] < 0$$
$$\frac{\partial V_c^i}{\partial \lambda_c^j} = 0 \qquad \forall\ i \neq j$$

which means that the Jacobian matrix of the utility is negative definite. It then follows from Rosen (1965, Theorem 8) that (**??**) always has a unique sectoral equilibrium in each city c.

B Proof outline of argmax$V^i(x) = 0$ or $1/2C$

In symmetric equilibrium $\lambda_c^i = a^i\mu/C$ for all c and i, the indirect utility of a worker in industry i at location x is given by

$$V^i(x) = \frac{\alpha^i a^i A_1}{2\mu\sigma(\sigma-1)(1+\varphi^i)\log\varphi^i\left(\phi^i\right)^x} + \text{constant}$$

where

$$\begin{aligned} A_1 \equiv\ & 2a^i\mu\sigma\left(1+\varphi^i\right)\left(\phi^i\right)^x \log\frac{\mu\left(1-\left(\varphi^i\right)^{C/2}\right)\left(\left(\phi^i\right)^{2x}+\varphi^i\right)}{\left(1-\varphi^i\right)\left(\phi^i\right)^x} \\ & -(1-\mu)(\sigma-1)\left(1-\left(\varphi^i\right)^2\right)\left[2\log\phi^i x\left(\phi^i\right)^{2x} + \left(\left(\phi^i\right)^{2x}+\varphi^i\right)\log\frac{1+\varphi^i}{\left(\phi^i\right)^{2x}+\varphi^i}\right] \\ & -\varphi^i\log\phi^i x\left[(\sigma-1)\left(1+\left(\phi^i\right)^{2x}-\mu+\left(\phi^i\right)^{2x}\mu+\left(1+\left(\phi^i\right)^{2x}+\mu-\left(\phi^i\right)^{2x}\mu\right)\phi^i\right)\right] \end{aligned}$$

Since $V^i(x) = V^i(1/C - x)$ holds for all x, we focus on the interval of $x \in [0, 1/2C]$.

We have

$$\frac{\mathrm{d}^4 V^i(x)}{\mathrm{d}x^4} = A_2\left[(z-1)\left(z^4+28z^3+38z^2+28z+1\right)-12z(z+1)^3\log z\right]$$

where $A_2 > 0$ and $z \equiv \left(\phi^i\right)^{2x-1/C}$. Examining $\mathrm{d}^4 V^i(x)/\mathrm{d}x^4$, we can show that $\mathrm{d}^3 V^i(x)/\mathrm{d}x^3$ is minimized at $x = 0$ or $1/2C$ for $x \in [0, 1/2C]$. We can then show $\mathrm{d}^3 V^i(0)/\mathrm{d}x^3 > 0$ and $\mathrm{d}^3 V^i(1/2C)/\mathrm{d}x^3 > 0$, and hence $\mathrm{d}^3 V^i(x)/\mathrm{d}x^3 > 0$ for $x \in [0, 1/2C]$. We can also show $\mathrm{d}^2 V^i(0)/\mathrm{d}x^2 < 0$, $\mathrm{d}V^i(0)/\mathrm{d}x < 0$, and $\mathrm{sgn}\left(\mathrm{d}^2 V^i(1/2C)/\mathrm{d}x^2\right) = \mathrm{sgn}\left(\mathrm{d}V^i(1/2C)/\mathrm{d}x\right)$. It follows that $V^i(x)$ is convex in the interval of $[0, 1/2C]$.

C Proof of Lemma 3

(i) From Bellman (1970, pp.242-243), the real parts of eigenvalues of the circulant Jacobian matrix are given by

$$z_c^i = \sum_{d=1}^{C} \frac{\partial J_d}{\partial \lambda_C^i} \cos\frac{2\pi cd}{C} \qquad \text{for } c = 1, 2, \ldots, C$$

Since the Jacobian at symmetric equilibrium is

$$\frac{\partial J_d}{\partial \lambda_C^i} = \frac{\mu}{C}\left(\frac{\partial V^i(d/C)}{\partial \lambda_C^i} - V^i(1) - \sum_e \frac{\partial V^i(e/C)}{\partial \lambda_C^i}\right)\Bigg|_{\lambda_c^i=\lambda^{i*}}$$

we have

$$z_c^i = \frac{\mu}{C}\sum_d \frac{\partial V^i(d/C)}{\partial \lambda_C^i}\cos\frac{2\pi cd}{C}\Bigg|_{\lambda_c^i=\lambda^{i*}} \tag{18}$$

The C-symmetric equilibrium in industry i is stable if (18) is negative for all $c = 1, 2, \ldots, C$.

It is straightforward that

$$\begin{aligned}\frac{\partial V^i(c/C)}{\partial \lambda_C}\Bigg|_{\lambda_c^i=\lambda^{i*}} &= \frac{\alpha^i}{\sigma^i\lambda^{i*}}\left[\frac{\left(2\sigma^i-1\right)\phi^i(x_c,1)}{(\sigma^i-1)\Phi^i} - \frac{\mu}{C\left(\Phi^i\right)^2\lambda^{i*}}\sum_d \phi^i(x_c,x_d)\,\phi^i(x_d,1)\right.\\ &\quad \left. -\frac{1-\mu}{\lambda^{i*}}\int_0^1 \frac{\phi^i(x_c,x)\,\phi^i(x,1)}{\left(\sum_d \phi^i(x,x_d)\right)^2}\mathrm{d}x\right]\end{aligned}$$

Thus, z_c^i in (18) is evaluated by plugging the RHS of this expression. It is readily shown that the denominator of z_c^i is positive, the numerators of z_{2c}^i and z_{2c-1}^i are increasing (resp. decreasing) in c for all $1 \le c \le C/4$ (resp. $C/4+1 \le c \le C/2$), and $z_{C/2}^i > z_{C/2\pm1}^i$ holds. That is, $z_{C/2}^i$ the largest, and hence the symmetry breaking condition is reduced to $\max_i z_{C/2}^i = 0$, or equivalently, $\tau = \tau_1(C)$. Hence, the stability condition is given by $\tau > \tau_1(C)$, and the instability condition is given by $\tau < \tau_1(C)$.

D Implicit functions of the boundaries

Straightforward calculations yield the boundaries between bifurcation and symmetry breaking:

$$\begin{cases} \mu = \widetilde{\mu}_{12}(\varphi) = \frac{(1+\varphi)^2\left[3-3\varphi^2+\left(1+4\varphi+\varphi^2\right)\log\varphi\right]}{3(1-\varphi^2)(1+\varphi)^2+(1+8\varphi+4\varphi^2+12\varphi^3-\varphi^4)\log\varphi} \\ \sigma = \widetilde{\sigma}_{12}(\varphi) = \frac{2\varphi\left[(1-\varphi)\left(1+10\varphi+\varphi^2\right)+6\varphi(1+\varphi)\log\varphi\right]}{(1-\varphi)(3+11\varphi+29\varphi^2+5\varphi^3)+(1+\varphi)(1+5\varphi+17\varphi^2+\varphi^3)\log\varphi} \end{cases}$$

for $C = 2$ and

$$\begin{cases} \mu = \widetilde{\mu}_{1}(\varphi) = \frac{(1+\varphi)^2\left[3-3\varphi^2+\left(1+4\varphi+\varphi^2\right)\log\varphi\right]}{3(1-\varphi^2)(1+\varphi)^2+(1+\varphi^2)(1+10\varphi+\varphi^2)\log\varphi} \\ \sigma = \widetilde{\sigma}_{1}(\varphi) = \frac{4\varphi\left[(1-\varphi)\left(1+10\varphi+\varphi^2\right)+6\varphi(1+\varphi)\log\varphi\right]}{(1-\varphi)(3+16\varphi+58\varphi^2+16\varphi^3+3\varphi^4)+(1+\varphi)(1+6\varphi+34\varphi^2+6\varphi^3+\varphi^4)\log\varphi} \end{cases}$$

for $C = 2^2, 2^3, \ldots, 2^k$. The boundary between existence and nonexistence of symmetric equilibrium is:

$$\begin{cases} \mu = \widetilde{\mu}_{2}(\varphi) = \frac{\varphi^{\frac{1}{2}}(1+\varphi)\log\varphi\left[(1+\varphi)\log\varphi+2\log\frac{2}{1+\varphi}\right]+2\left(1-\varphi^{\frac{1}{2}}\right)^2\left(\varphi\log\frac{2\varphi}{1+\varphi}-\log\frac{2}{1+\varphi}\right)}{2\varphi^{\frac{1}{2}}(1+6\varphi+\varphi^2)\log\varphi\,\log\frac{2\varphi^{\frac{1}{2}}}{1+\varphi}+2\left(1-\varphi^{\frac{1}{2}}\right)(1-\varphi^2)\left(\varphi^{\frac{1}{2}}\log\frac{2\varphi}{1+\varphi}-\log\frac{2}{1+\varphi}\right)} \\ \sigma = \widetilde{\sigma}_{2}(\varphi) = \frac{B_1}{B_2} \end{cases}$$

for $C = 2^1, 2^2, \ldots, 2^k$, where

$$\begin{aligned} B_1 &\equiv 2\varphi^{\frac{1}{2}}\left\{2(1-\varphi)\left[4\varphi\log(1+\varphi)-1-\varphi+2(1-\log 2)\varphi^{\frac{1}{2}}\right]-\left(1-\varphi^{\frac{1}{2}}\right)\left(1-\varphi^{\frac{1}{2}}+5\varphi-\varphi^{\frac{3}{2}}\right)\log\varphi\right\} \\ B_2 &\equiv \varphi^{\frac{1}{2}}\log\varphi\left[2(2+\varphi)\varphi\log 2-2\left(1-\varphi^{\frac{1}{2}}\right)\left(1-2\varphi^{\frac{1}{2}}+6\varphi-2\varphi^{\frac{3}{2}}+\varphi^2\right)+2\log 2+(1+\varphi)^2\log\varphi\right] \\ &\quad +2\log(1+\varphi)-(1-\varphi)\left[4\varphi^{\frac{1}{2}}\left(1-2\varphi^{\frac{1}{2}}+\varphi-(1+\varphi)\log 2\right)+2\left(1+6\varphi+\varphi^2\right)\log 2\right] \\ &\quad -2\varphi^{\frac{1}{2}}\left[2-5\varphi^{\frac{1}{2}}+5\varphi^{\frac{3}{2}}-2\varphi^2+\varphi^{\frac{5}{2}}+(1+\varphi)^2\log\varphi\right]\log(1+\varphi) \end{aligned}$$

References

[1] Alao, N., M.F. Dacey, O. Davies, K.G. Denike, J. Huff, J.B. Parr and M.J. Webber (1977) *Christaller Central Place Structures: An Introductory Statement.* Northwestern University Studies in Geography 22, Evanston.

[2] Christaller, W. (1933) *Die Zentralen Orte in Süddeutschland.* Jena: Gustav Fischer Verlag. English translation: *The Central Places of Southern Germany.* Englewood Cliffs (N.J.): Prentice-Hall, 1966.

[3] Eaton, B.C. and R.G. Lipsey (1982) An economic theory of central places. *Economic Journal* 92, 56-72.

[4] Fujita, M., P. Krugman and T. Mori (1999) On the evolution of hierarchical urban systems. *European Economic Review* 43, 209-251.

[5] Henderson, J.V. (1974) The sizes and types of cities. *American Economic Review* 64, 640-56.

[6] Henderson, J.V. (1988) *Urban Development. Theory, Fact and Illusion.* Oxford: Oxford University Press.

[7] Krugman, P. (1991) Increasing returns and economic geography. *Journal of Political Economy* 99, 483-499.

[8] Krugman, P. (1993) On the number and location of cities. *European Economic Review* 37, 293-298.

[9] Lösch, A. (1940) *Die Räumliche Ordnung der Wirtschaft.* Jena: Gustav Fischer. English translation: *The Economics of Location.* New Haven: Yale University Press (1954).

[10] Marsall, J.U. (1989) *The Structure of Urban Systems.* Toronto: The University of Toronto Press.

[11] Martin, P. and C.A. Rogers (1995) Trade effects of regional aid. In: Baldwin R., P. Haaparanta and J. Kiander (eds.) *Expanding Membership of the European Union.* Cambridge: Cambridge University Press, 166– 188.

[12] Ottaviano, G.I.P., T. Tabuchi and J.-F. Thisse (2002) Agglomeration and trade revisited. *International Economic Review* 43, 409-436.

[13] Pflüger, M.P. (2004) A simple, analytically solvable Chamberlinian agglomeration model. *Regional Science and Urban Economics* 34, 565-573.

[14] Quinzii, M. and J.-F. Thisse (1990) On the optimality of central places. *Econometrica* 58, 1101-1119.

[15] Rasband, S.N. (1990) *Chaotic Dynamics of Nonlinear Systems.* New York: Wiley.

[16] Rosen, J.B. (1965) Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica* 33, 520-534.

[17] Scotchmer, S. (2002) Local public goods and clubs. In: A. Auerbach and M. Feldstein (eds.) *Handbook of Public Economics* Volume IV. 1997-2042. Amsterdam: North-Holland.

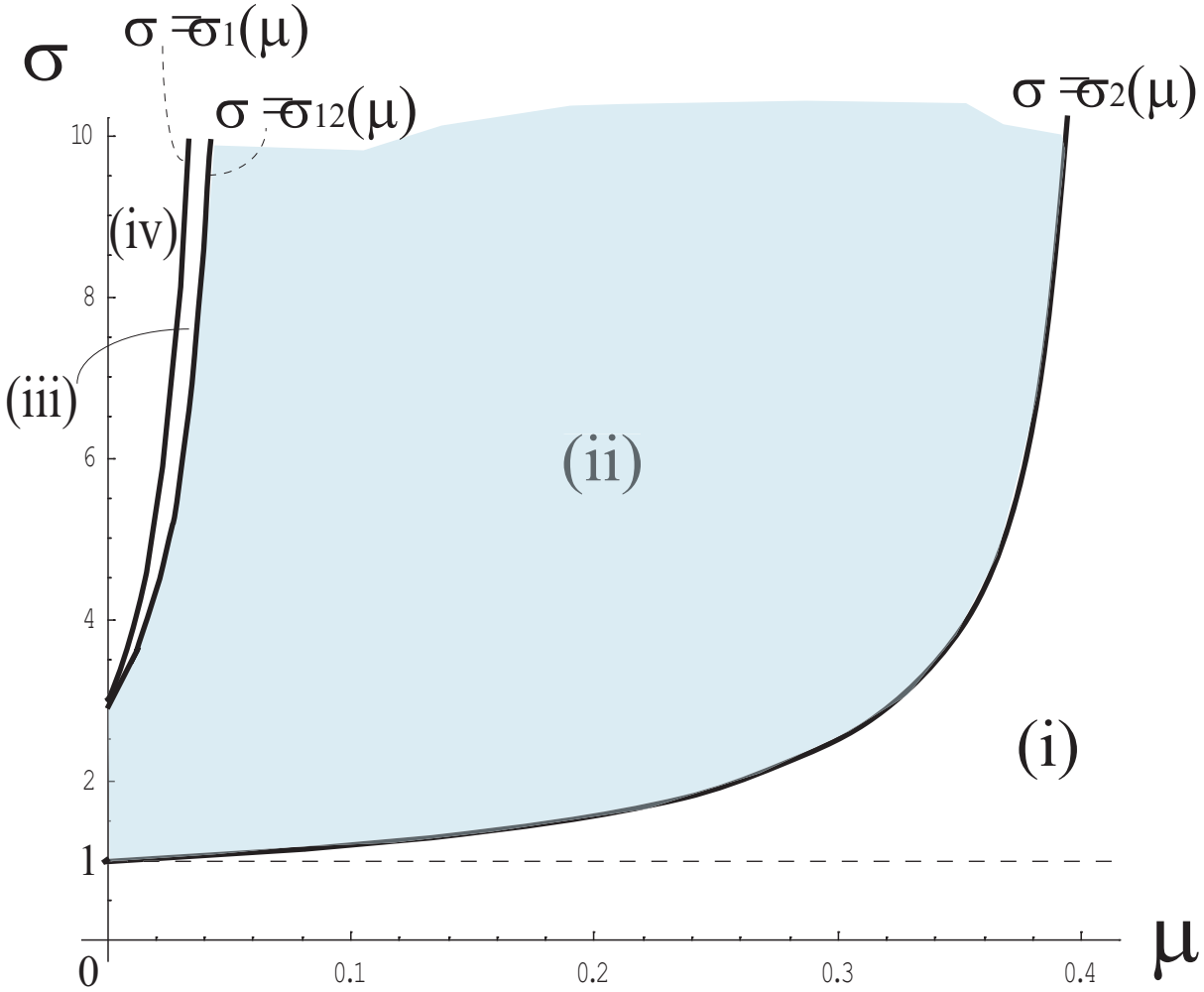

Figure 1: Bifurcation domain (ii) in Proposition 2

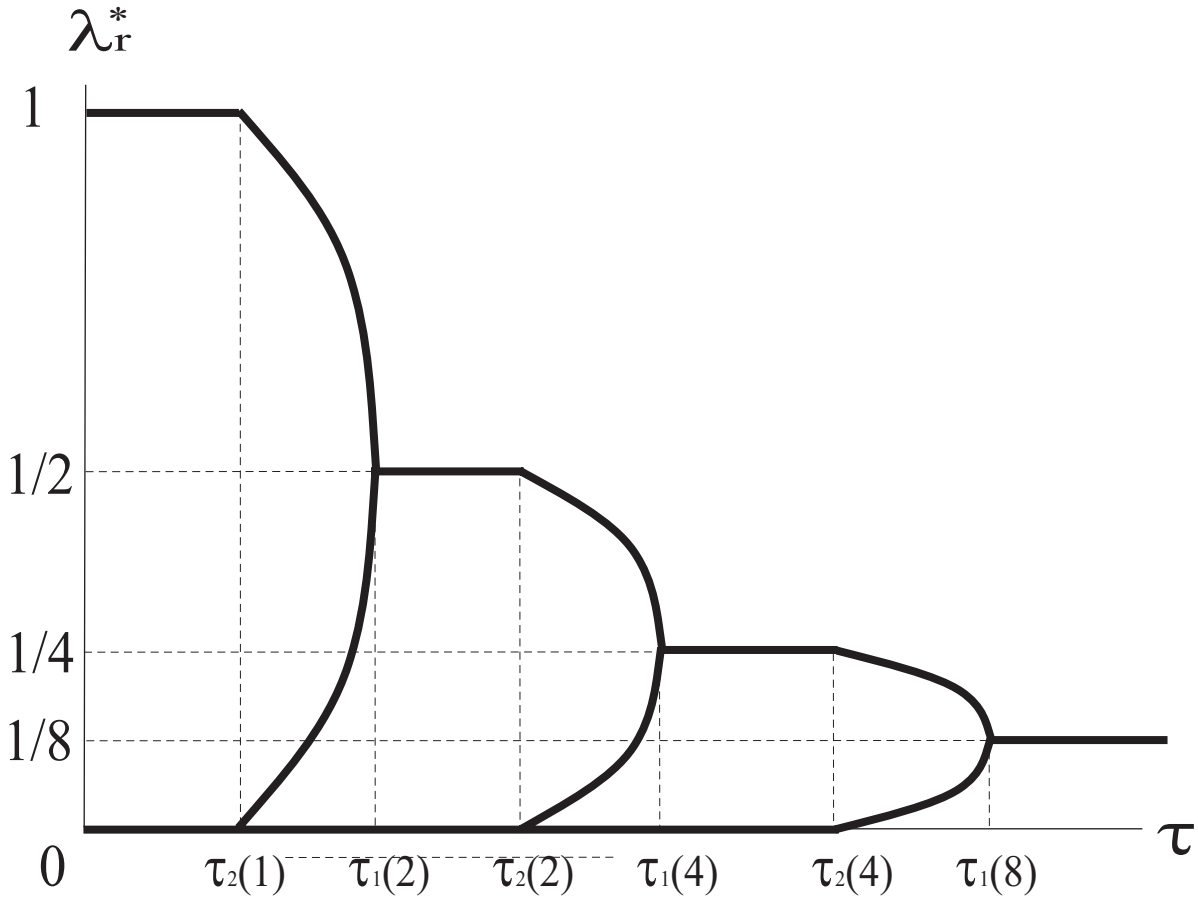

Figure 2: Evolutionary process for transport cost changes

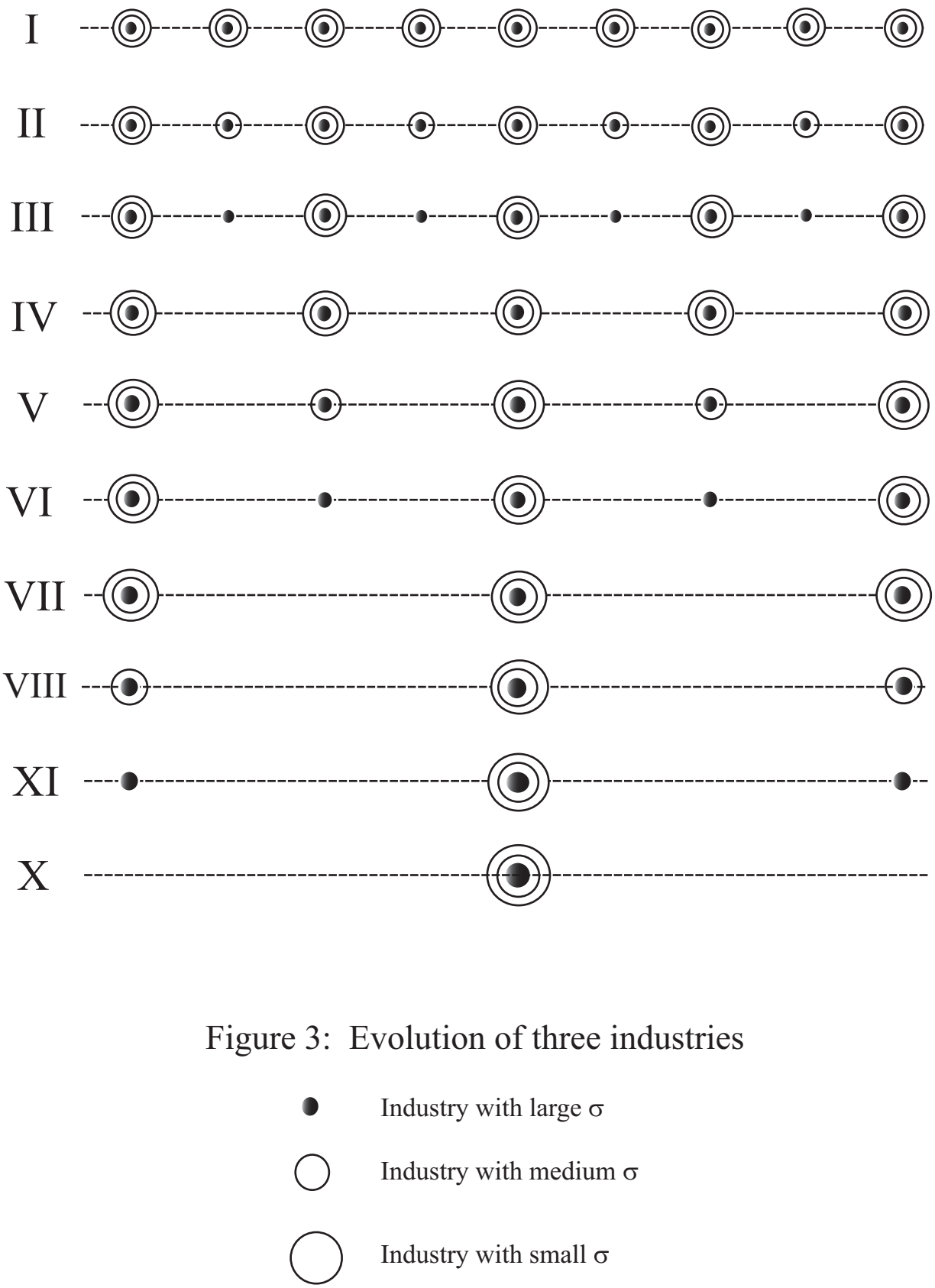

Figure 3: Evolution of three industries