Evolution and Local Interaction with One Forward-Looking Player*

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Abstract

This paper studies a simple evolutionary model of local interaction with one forward-looking player and many myopic players. Myopic players are positioned along a circle, and each myopic player interacts with his two immediate neighbors and the forward-looking player. The stage game played during each period is a 2×2 symmetric coordination game in which each myopic player plays a strategy identical to that played by two or three of his neighbors in the previous period. If the forward-looking player is sufficiently patient, efficient equilibrium is uniquely selected as the long-run stochastically stable state. We derive the waiting time for reaching the equilibria, which strengthen our results. Furthermore, if a cluster of myopic players does not interact with the forward-looking player, the coexistence of conventions can be the long-run stochastically stable state. *JEL Classification:* C73

Keywords: stochastic stability, local interaction, forward looking player, efficient equilibrium, waiting time, coexistence of conventions

1 Introduction

In standard evolutionary models with ε-noise, the behavior of players is assumed to be the best response to the distribution of the opponents' play in the previous

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period. Kandori, Mailath and Rob (1993) and Young (1993) showed that under this assumption, if players are randomly paired in each period to play the 2×2 symmetric coordination game, the risk-dominant equilibrium has the larger of the two basins of attraction. Therefore, it is selected in the limit of the invariant distribution as the mutation rate tends to zero (long-run stochastically stable). Such a risk-dominant equilibrium may indeed be Pareto-dominated by the other equilibrium. Therefore, in the event of a conflict between the risk-dominant considerations and Pareto-efficient considerations, the risk-dominant but inefficient equilibrium would be the long-run stochastically stable one.

Ellison (1993) explored a matching rule in which players interact with a small group of close friends, neighbors, or colleagues. As a special case, Ellison considered a network wherein players are positioned along a circle, and each player is only matched with one of his two immediate neighbors. Ellison showed that according to this matching rule, the risk-dominant equilibrium is long-run stochastically stable, and the evolutionary system can accelerate convergence to this equilibrium.

This paper considers a situation wherein some players take into consideration the future in order to decide the current action to be taken in contrast to the situation wherein all players behave myopic best response. As a special case, we focus on the situation wherein only one player is forward-looking.

This paper considers the local interaction structure, which is similar to that in Ellison (1993), wherein many but finite myopic players are positioned along a circle, and each myopic player interacts with his two immediate neighbors and the forward-looking player (the forward looking player is thus linked with all the myopic players)[1]. Figure 1 shows the resulting network with eight myopic players. The players are randomly paired in each period to play the 2×2 symmetric coordination game in which the risk-dominant equilibrium is Pareto-dominated by another equilibrium, and each myopic player on the circle would play the same strategy that two or three of his neighbors played in the previous period, for example, as shown in figure 2[2].

Example: Consider a small town with a grocery store and many (but finite) residents. The residents are positioned along a circle such that each of them is friends with his two immediate neighbors, and every resident is a customer of the

[1]Möbius (2001) considered the same network and stage game as in this paper, wherein the player who interacts with all the other players is not forward-looking, but adopts two pure strategies simultaneously.

[2]Suppose that a player on a circle has two neighbors who played A in the previous period. If the player plays A today, he earns an expected payoff of $\frac{2}{3} \times 1 + \frac{1}{3} \times 0 = \frac{2}{3}$, which is greater than the expected payoff of $\frac{2}{3} \times (-2) + \frac{1}{3} \times 2 = -\frac{2}{3}$ from playing B. Now suppose that a player on a circle has only one neighbor who played A in the previous period. If the player plays A today, he earns an expected payoff of $\frac{1}{3} \times 1 + \frac{2}{3} \times 0 = \frac{1}{3}$, which is smaller than the expected payoff of $\frac{1}{3} \times (-2) + \frac{2}{3} \times 2 = \frac{2}{3}$ from playing B. Hence, in this stage game, each myopic player along the circle would play the same strategy that two or three of his neighbors played in the previous period.

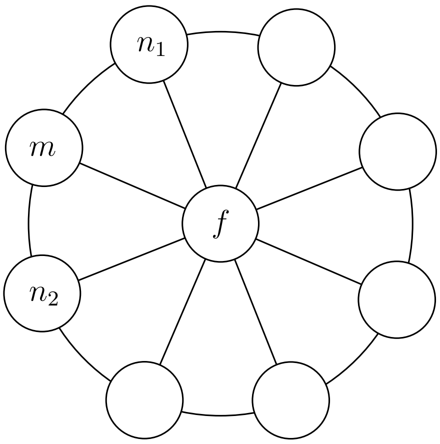

Figure 1: Network – Connected line between players means that they are neighbors each other. Myopic player m is neighbors with player n_1, n_2 and f (forward looking player).

	A	B
A	$1, 1$	$0, -2$
B	$-2, 0$	$2, 2$

Figure 2: The 2×2 symmetric coordination game. (A, A) is risk-dominant and (B, B) is Pareto-efficient. Each myopic player along the circle would play the same strategy that two or three of his neighbors played in the previous period.

store. In each period, every resident exchanges messages with either his friends or the store. There are two ways to communicate. A resident may either write a letter to a friend or walk to the store to place an order, and the store presents him handbills. Alternately, these interactions may be carried out over the Internet. Since online interactions are quick and costless, they are considered to be Pareto-efficient. Since one has to purchase a personal computer in order to exchange messages over the Internet, the loss incurred in purchasing a personal computer is greater than that incurred in purchasing a sheet of paper to write a letter on when one's strategy is different from the opposite. Hence, offline interactions are considered to be risk-dominant.

A forward-looking player discounts the future at a rate $\delta \in (0,1)$. Suppose that he has an opportunity to revise his choice in period t. He observes his neighbors' (all myopic players) distribution of play in period $t-1$, predicts the future state resulting from his current and future plays, and then decide to play the current action in order to maximize the summation of the expected present discounted payoffs of the stage game.

Let $\{A, B\}$ be the set of strategies for the stage game, (A, A) be the risk-dominant equilibrium, and (B, B) be the Pareto-efficient equilibrium. We refer to strategies A and B as risk-dominant and efficient strategies, respectively. Since each myopic player plays the same strategy that two or three of his neighbors played in the previous period, if the choice of the myopic player is different from that of his myopic neighbor's, the choice of the forward-looking player can manipulate this myopic player's choice. Thus, if some myopic players played B in the previous period, a forward-looking player, by continuing to play B, can lead the society to a state wherein all players would play B in the future. Thus, even if only one myopic players played B in the previous period, the forward-looking player will continue to play B in order to earn an efficient payoff in the future, provided he is sufficiently patient (δ is sufficiently large).

The main result of this paper is that the evolutionary system with a forward-looking player uniquely selects the Pareto-efficient equilibrium as the long-run stochastically stable state (Theorem 1 in Section 3). As per the abovementioned argument, if one player on the circle played B in the previous period, the forward-looking player would also play B in the current period. This behavior easily leads the state to a Pareto-efficient equilibrium. Hence, the efficient equilibrium has the larger of the two basins of attraction and, therefore, is selected in the limit of the invariant distribution as the mutation rate tends to zero.

Theorem 1 also shows that the system without a forward-looking player (all players in the network behave myopically) selects both the risk-dominant and the Pareto-efficient equilibria in the long run, a result that weakens the main result. However, using the technique of waiting time for reaching these equilibria, we claim that in the case of a fixed small mutation rate and large population size, the risk-dominant equilibrium is more persistent than the Pareto-efficient equilibrium in the system without a forward-looking player, whereas the reverse is true for the system with the forward-looking player (Theorem 2 in Section 4). Möbius (2000) argued that in the case of a large population, stochastic stability can fail to predict the long-run behavior of the system because it ignores a large amount of information regarding the unperturbed process, particularly in the local interaction structure. The system without a forward-looking player is the case. The unperturbed process in this system tends to push the wide extent of the states to the risk-dominant equilibrium and, therefore, this equilibrium is more persistent in the case of a large population.

This paper also presents arguments on the issue of coexistence of conventions. Berninghaus and Schwalbe (1996), Goyal and Janssen (1997), Morris (2000), and Sugden (1995) discussed which properties of the interaction structure make the coexistence of conventions more or less likely. This paper provides an example of a network with one forward-looking player, such that the state wherein conventions coexist would be long-run stochastically stable (Theorem 3 in Section 5). We consider a network wherein myopic players are positioned around a circle, and the cluster of players interacts only with its two immediate neighbors (and the remaining cluster of players interacts with its two immediate neighbors and the forward-looking player). As stated in Theorem 1, efficient strategies tend to prevail in regions with a forward-looking player. On the other hand, as seen in Ellison's (1993) model, risk-dominant strategies tend to prevail in regions without a forward-looking player. Therefore, this system can easily lead to a state wherein conventions coexist.

The following is a brief comment on some related literature pertaining to efficient equilibrium selection. Bergin and Lipman (1996) introduced the concept of state-dependent mutation. It appears reasonable that mutation rates might differ depending on the transition involved. Bergin and Lipman showed that for a given population size the evolutionary process selects any strict Nash equilibrium. Efficient equilibrium is selected if experimentation (mutation) by players is rare when they earn the efficient payoff. This paper does not assume state-dependent mutation. The mutation rate of each player is considered independent across players and periods.

Robson and Vega-Redondo (1996) studied a literally random matching mechanism, combined with a strategy adjustment process based on the realized average performance of each strategy. In the case of a large population, the efficient equilibrium is selected. In this paper, strategy adjustment by each player is based on the expected payoff of each strategy, which is a familiar assumption in evolutionary game theory.

Ely (2002) considers players' mobility. The players can select the location in which he plays the stage game as well as the strategy. As mentioned above, the network is considered to be exogenous in this paper, and hence, we do not assume

	A	B
A	a, a	d, c
B	c, d	b, b

Figure 3: The 2×2 symmetric coordination game.

the players' mobility.

This paper is organized as follows. Section 2 describes the basic game structure and the behavior of each player. Section 3 describes the long-run behavior of the system (Theorem 1). Section 4 considers the waiting time when the population size is large (Theorem 2). The issue regarding the coexistence of conventions is discussed in Section 5 (Theorem 3). Section 6 presents a conclusion for this study.

2 Model

Let $\mathcal{I} = \{0, 1, \ldots, N\}$ represent the set of players (N is large but finite) and $\mathcal{I}_c = \{1, \ldots, N\} \subset \mathcal{I}$. Players in $\mathcal{I}_c$ are positioned around a circle, and each $i \in \mathcal{I}_c$ interacts with his two immediate neighbors and player 0. Thus, player 0 is linked to all the players in $\mathcal{I}_c$. Formally, let $\Phi(i)$ ($i \in \mathcal{I}$) be the set of neighbors of player i. Then

$$\Phi(0) = \mathcal{I}_c,$$
$$\Phi(i) = \{0, i-1, i+1\} \quad \forall i \in \mathcal{I}_c,$$

($i - 1 = N$ if $i = 1$, and $i + 1 = 1$ if $i = N$). We consider two networks, $G_f(N+1)$ and $G_m(N+1)$ in order to investigate the role of the forward-looking player: In $G_f(N+1)$, player 0 is forward-looking and the players in $\mathcal{I}_c$ are myopic, and in $G_m(N+1)$, all the players behave myopically. Time is divided into discrete periods. In each period $t = 0, 1, 2, \ldots$, each player in $\mathcal{I}_c$ is randomly paired with one of his neighbors to play the 2×2 symmetric coordination game in which all the payoffs are real numbers, as shown in figure 3.

For the purpose of convenience, it is assumed that player 0 can simultaneously match with multiple players and earn an average payoff, so that each player in $\mathcal{I}_c$ meets his neighbors with equal probability. It is also assumed that $a > c$ and $b > d$; thus, (A, A) and (B, B) are both Nash equilibria, and that $b > a$; thus, (B, B) is the efficient equilibrium. Let q^* be the probability assigned to the strategy A in mixed strategy equilibrium, i.e., $q^* \equiv \frac{b-d}{(a-c)+(b-d)}$. The following is an important assumption of the model presented in this paper.

Assumption 1. $q^* \geqslant \dfrac{1}{3}$.

The importance of this assumption is explained in the next subsection. It is assumed that $q^* \in [\frac{1}{3}, \frac{1}{2})$ so that the efficient equilibrium is not risk-dominant.

In each period, every player has the opportunity to revise his current strategy choice with probability $r \in (0, 1)$ and retains his strategy choice with probability $1 - r$. The probability r is independent across players and periods. When the opportunity for strategy revision arrives, the player chooses an action from his action set $\{A, B\}$ according to his learning rule, which is described in the following two subsections.

At the beginning of a period, and after choosing an action, each player takes a draw from an independent, identically distributed Bernoulli random variable. With probability $\varepsilon \in (0, 1)$, this player is a mutant and chooses an action at random with 50-50 probability. With probability $1 - \varepsilon$, this player does not experience any mutation.

The state of the system may be defined by an $(N+1)$-tuple $z = (a_0, a_1, \ldots, a_N) \in \mathcal{Z} = \{A, B\}^{N+1}$, indicating the action taken by each player.

The time schedule is as follows. At the beginning of the period, the player with the opportunity to revise his strategy chooses an action, and each player mutates with probability ε. The players who are randomly paired with their neighbors then play the abovementioned stage game and earn a payoff. If a player is not matched with anyone, he earns a payoff of 0.

2.1 Behavior of Myopic Players

The behavior of a myopic player is assumed to be the best response to the distribution of his neighbors' play in the previous period, which is widely used in evolutionary game theory literature (See, for example, Kandori, Mailath and Rob, 1993; Young, 1993; Ellison, 1993). Let q_i $(i \in \mathcal{I})$ be the fraction of player i's opponents who played strategy A in the previous period. If $q_i > q^*$, player i plays A, and if $q_i < q^*$, he plays B. In the knife-edge case where $q_i = q^*$, it is assumed that player i plays B. Since we consider the case $q^* \in [\frac{1}{3}, \frac{1}{2})$, we can describe the behavior of myopic players in $\mathcal{I}_c$ in greater detail; Each myopic player plays A (B) if two or three of his neighbors played A (B) in the previous period. On the basis of Assumption 1 and $q^* < \frac{1}{2}$, if player $i - 1$'s strategy choice is different from that of player $i + 1$'s, then player 0's choice has a great influence on player i's choice–player i chooses the same strategy as player 0.

2.2 Behavior of the Forward-Looking Player

The forward-looking player discounts the future at a rate $\delta \in (0, 1)$. Suppose that he has an opportunity to revise his choice in period t. He observes his neighbors' (all myopic players) distribution of the play in the previous period, predicts the future state resulting from his current and future plays, and then decides to play the current action in order to maximize the summation of the expected present discounted payoff streams

$$\pi_t = \sum_{s=t}^{\infty} \delta^{s-t} u_s,$$

where u_s denotes the expected payoff in the periods $s = t, t+1, \ldots$.

The strategy of the forward-looking player, $\sigma : \mathcal{Z} \to \{A, B\}$, is the contingent plan that specifies the action chosen by him for each state in the previous period. Note that the plan does not depend on the period t in which it is his turn to move but on the state in the previous period. For example, suppose that the states realized in periods T and T' are identical. In that case, the expected present discounted payoffs in periods $T+1$ and $T'+1$ from the same course of current and future play will be identical. Therefore, a plan should be developed for each state in the previous period.

In order to predict the future state of the world, the forward-looking player requires some knowledge regarding the society; he is assumed to have knowledge on the following:

- the stage game,
- the network,
- neighbors' behavior (the best response to the opponent's play in the previous period),
- the opportunity for strategy revision arriving with probability r,
- and each player mutating with probability ε (including his own mutation).

We are required to consider the behavior of the forward-looking player both with and without noise. If the mutation rate ε is large, the behavior of the forward-looking player with noise and without noise may be different because his prediction of the future state will differ in two cases. However, only the case wherein each player seldom mutates is of interest to us; hence, we focus on the case in which ε is sufficiently small. Let $\hat{z}$ denote the state in the previous period. The following proposition describes the optimal strategy $\sigma^*(\hat{z})$.

Proposition 1. *Let $q^* \in [\frac{1}{3}, \frac{1}{2})$ and $N \geqslant 3$.*

1. *(Without noise) There exists $\bar{\delta} \in (0, 1)$ such that for all $\delta \in (\bar{\delta}, 1)$,*

$$\sigma^*(\hat{z}) = \begin{cases} A & if\ \hat{z} = (A, A, \ldots, A)\ or\ \hat{z} = (B, A, \ldots, A) \\ B & otherwise. \end{cases}$$

2. *(With noise) Given that* $\delta \in (\bar{\delta}, 1)$*, there exists* $\bar{\varepsilon}(\delta) > 0$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon}(\delta))$*, the above strategy is still optimal.*

Proof. See Appendix A. □

In other words, player 0 plays B if at least one of the myopic players played B in the previous period. The intuition is as follows. A myopic player plays B if two or three of his neighbors played B in the previous period. Thus, the myopic player who plays A and has one myopic neighbor playing B will play B if player 0 plays B. Therefore, there exists a possibility that if player 0 continues to play B, the system moves from a state in which at least one of the myopic players plays B to one in which all players play B. Hence, if at least one of the myopic players played B in the previous period, a sufficiently patient forward-looking player would also play B in order to realize a higher expected payoff from the stage game in the future.

Hereafter, we choose δ and ε such that $\delta \in (\bar{\delta}, 1)$ and $\varepsilon \in (0, \bar{\varepsilon}(\delta))$. For states $z \in \mathcal{Z}$ and $z' \in \mathcal{Z}$, let $p_m(z, z', 0)$ be the probability that in $G_m(N+1)$, a single iteration of the myopic best response process changes the state of the system to z', given that the current state is z. Also, let $p_f(z, z', 0)$ be the probability that in $G_f(N+1)$ a single iteration of each player's learning rule changes the state of the system to z', given that the current state is z. The collections $\mathbf{P}_m(0) \equiv \{p_m(z, z', 0)\}_{z,z' \in \mathcal{Z}}$ and $\mathbf{P}_f(0) \equiv \{p_f(z, z', 0)\}_{z,z' \in \mathcal{Z}}$ along with the specification of the initial state are Markov processes on the state space $\mathcal{Z}$.

3 Long-run Behavior

This section includes the main result that with a sufficiently patient forward-looking player, efficient equilibrium is uniquely selected as the long-run stochastically stable state in which the process spends almost all its time as the mutation rate becomes small.

3.1 Limit Sets

Let $p_x^\tau(z, z', 0)$ be the probability that in $G_x(N+1)$ $(x = m, f)$, the process of each player following his learning rule changes the state of the system to z' in τ steps, given that the current state is z. In order to investigate the long-run behavior of the system, we first introduce the concept of limit sets that are formally defined as follows:

Definition 1. *A set* $\omega \subset \mathcal{Z}$ *is a limit set if, under* $\mathbf{P}_x(0)$ $(x = m, f)$*,*

1. $\forall z \in \omega, \forall z' \notin \omega, \quad p_x(z, z', 0) = 0$*, and*

2. $\forall z, z' \in \omega, \exists T > 0, \quad p_x^T(z, z', 0) > 0.$

The collection of all limit sets is denoted by Ω_x.

In other words, a limit set is a minimal set of states with the property that a process can lead into this set but out of it.

The following proposition computes the limit sets in our evolutionary system. $\vec{A}$ and $\vec{B}$ will be used to denote $(A, A, \ldots, A) \in \mathcal{Z}$ and $(B, B, \ldots, B) \in \mathcal{Z}$, respectively.

Proposition 2. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $N \geqslant 3$. *Then, for each* $x = m, f$, $\Omega_x = \{\{\vec{A}\}, \{\vec{B}\}\}$.

Proof. See appendix B. □

3.2 Long-run Stochastically Stable State

Let $p_x(z, z', \varepsilon)$ be the probability that in $G_x(N+1)$ $(x = m, f)$, the combination of each player's learning process and players' mutations change the state of the system to z', given that the current state is z. $\mathbf{P}_m(\varepsilon) \equiv \{p_m(z, z', \varepsilon)\}_{z,z' \in \mathcal{Z}}$ and $\mathbf{P}_f(\varepsilon) \equiv \{p_f(z, z', \varepsilon)\}_{z,z' \in \mathcal{Z}}$ are again Markov processes on the state space $\mathcal{Z}$. The probability distribution at time t over all states is represented by a $1 \times 2^{N+1}$ vector w_t. The evolution of the process is governed by

$$w_{t+1} = w_t \mathbf{P}_x(\varepsilon) \quad (x = m, f).$$

It should be noted that each element of $\mathbf{P}_x(\varepsilon)$ is strictly positive for $\varepsilon > 0$; therefore, from standard results on Markov processes, there exists a unique invariant distribution $\mu_x(\varepsilon)$ $(x = m, f)$, such that

$$\mu_x(\varepsilon) = \mu_x(\varepsilon)\mathbf{P}_x(\varepsilon).$$

We focus on the limit distribution μ_x^* $(x = m, f)$, defined as $\mu_x^* = \lim_{\varepsilon \to 0} \mu_x(\varepsilon)$. The probability assigned to state z by the distribution μ_x^* is denoted by $\mu_x^*(z)$.

Definition 2. *The state* $z \in \mathcal{Z}$ *is the long-run stochastically stable state if* $\mu^*(z) > 0$.

The proof of the main theorem is based on the mutation-counting argument. First, the cost of transition between two limit sets is defined. The cost of transition between two *states* is defined as the minimum number of mutations required to move from one state to another in a single step. Let $c(z, z')$ denote the cost of transition from $z \in \mathcal{Z}$ to $z' \in \mathcal{Z}$. Consider a finite sequence of distinct states $(z_0, z_1, \ldots, z_L)$. The cost of transition between two limit sets $\omega, \omega' \in \Omega$ is defined as

$$C(\omega, \omega') = \min_{\substack{z_0, z_1, \ldots, z_L \\ z_0 \in \omega \\ z_L \in \omega' \\ L}} \sum_{l=1}^{L} c(z_{l-1}, z_l),$$

where, for all z_l ($l = 0, 1, \ldots, L$), $z_l \notin \omega''$ ($\omega'' \in \Omega$, $\omega'' \neq \omega, \omega'$). In other words, the cost of transition between two limit sets is the minimum number of mutations required to move from one limit set to the other over time. From the definition of limit sets, it is evident that for all $\omega, \omega' \in \Omega$, $C(\omega, \omega') \geqslant 1$.

Given a finite set Ω and $\omega \in \Omega$, an ω-tree on Ω is a collection of directed branches (ω^0, ω^1) (ω^1 being the successor of ω^0), where with the exception of ω, each limit set has a unique successor and there are no closed loops. In other words, it is a tree directed into root ω. Let H_ω denote the set of all ω-trees. The following lemma is well-known (see, for example, Young, 1993; Kandori and Rob, 1995).

Lemma 1. *The set of long-run stochastically stable states is the solutions to*

$$\min_{\omega \in \Omega} \min_{h \in H_\omega} \sum_{(\omega', \omega'') \in h} C(\omega', \omega'').$$

In other words, the set of long-run stochastically stable states is the set of limit sets, which have the minimum sum of costs of branches in their possible trees. Since there are only two limit sets in our evolutionary system, $H_{\vec{A}}$ and $H_{\vec{B}}$ are singletons. Therefore, it is sufficient to compare $C(\vec{A}, \vec{B})$ and $C(\vec{B}, \vec{A})$ in order to investigate the long-run stochastically stable state. If $C(\vec{A}, \vec{B}) > C(\vec{B}, \vec{A})$, then $\vec{A}$ is the unique long-run stochastically stable state. If $C(\vec{B}, \vec{A}) > C(\vec{A}, \vec{B})$, then $\vec{B}$ is the unique long-run stochastically stable state. If $C(\vec{A}, \vec{B}) = C(\vec{B}, \vec{A})$, then both $\vec{A}$ and $\vec{B}$ are the long-run stochastically stable states.

Theorem 1. *Let $q^* \in [\frac{1}{3}, \frac{1}{2})$ and $N \geqslant 3$. Then,*

1. *In $G_m(N+1)$, both $\vec{A}$ and $\vec{B}$ are the long-run stochastically stable states.*
2. *In $G_f(N+1)$, $\vec{B}$ is the unique long-run stochastically stable state.*

Proof. See Appendix C. □

4 Large Population

As shown in Theorem 1 in the previous section, the efficient equilibrium would be long-run stochastically stable even if player 0 behaves myopically. This result weakens our main result that the efficient equilibrium would be uniquely long-run stochastically stable if player 0 was forward-looking. This section, however, claims using the technique of waiting time for reaching these equilibria, that for a sufficiently small but non-negligible mutation rate and large population, the system of $G_m(N+1)$ spends almost all its time around an inefficient (risk-dominant) equilibrium and that of $G_f(N+1)$, on the other hand, spends almost all its time around an efficient equilibrium.

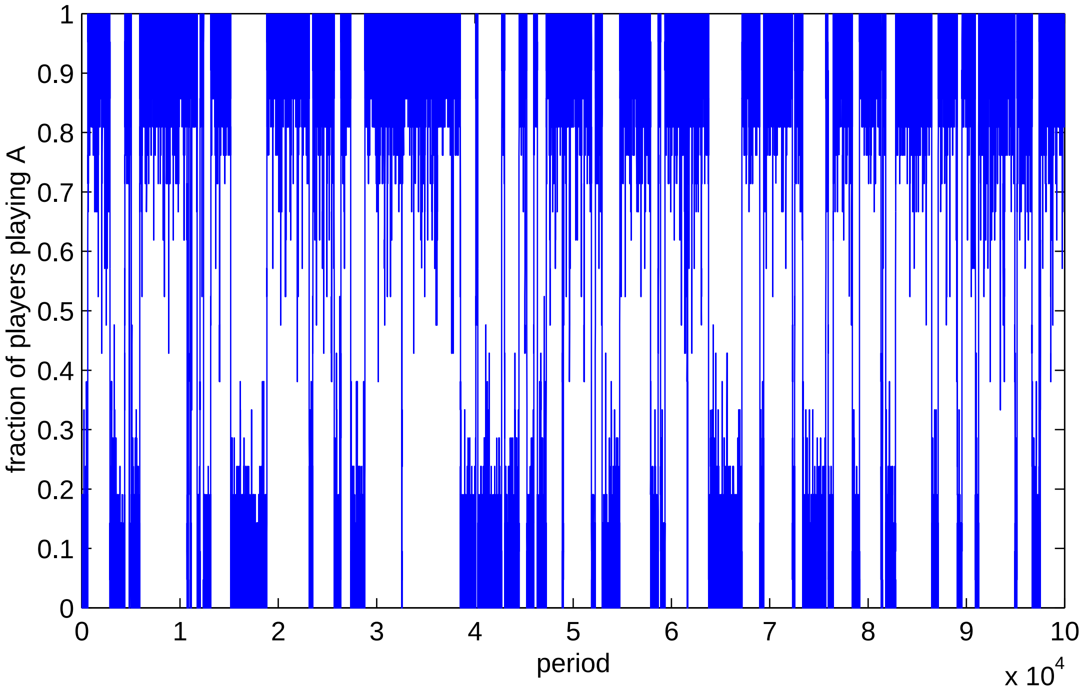

Figure 4: Simulation result of $G_m(N+1)$ —— $N = 20$ ($G_m(21)$). $r = 0.5$, $\varepsilon = 0.05$, $q^* = 0.4$, initial state $= \vec{B}$ (fraction of players playing $A = 0$).

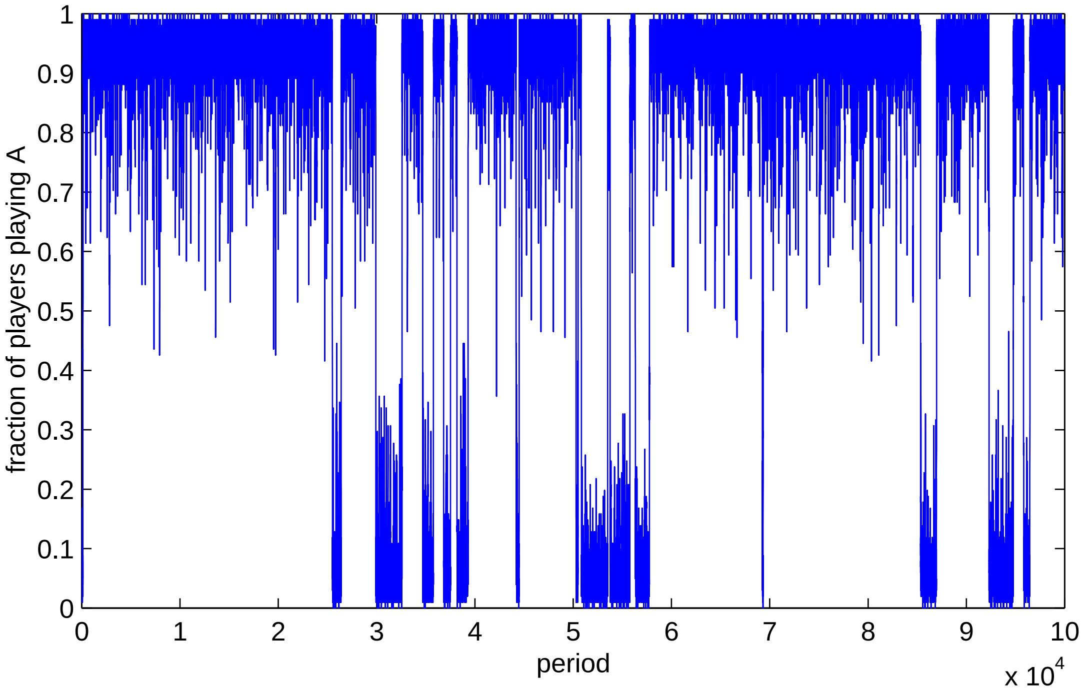

Figure 5: Simulation result of $G_m(N+1)$ —— $N = 100$ ($G_m(101)$). $r = 0.5$, $\varepsilon = 0.05$, $q^* = 0.4$, initial state $= \vec{B}$ (fraction of players playing $A = 0$).

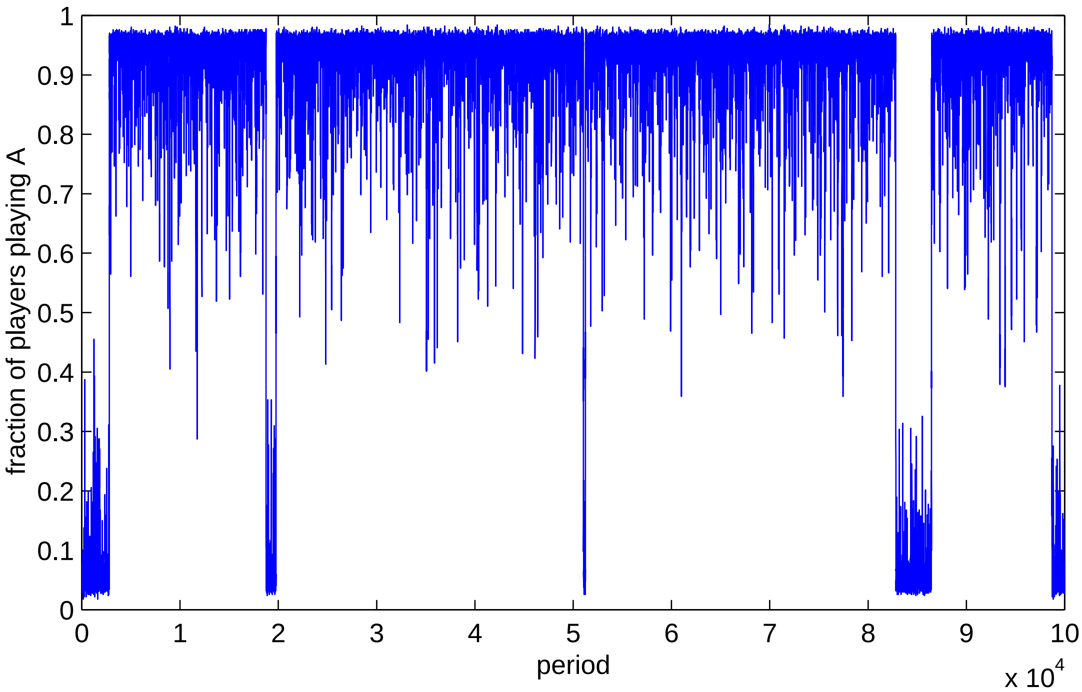

Figure 6: Simulation result of $G_m(N+1)$ —— $N = 500$ ($G_m(501)$). $r = 0.5$, $\varepsilon = 0.05$, $q^* = 0.4$, initial state $= \vec{B}$ (fraction of players playing $A = 0$).

Figures 4 - 6 illustrate the simulation results for the behavior of the system of $G_m(N+1)$ for 100,000 period, where $r = 0.5$, $\varepsilon = 0.05$, $q^* = 0.4$, initial state is $\vec{B}$ (all players play B), and the population size varies. It is likely that the larger the population size, the longer the system remains in the inefficient (risk-dominant) equilibrium (the fraction of players playing A is 1). This observation requires us to conduct a more careful analysis in the case of a large population.

4.1 Waiting Times

Let $W_x(z, Z', N)$ be the expected waiting time in $G_x(N+1)$ ($x = m, f$) until a state belonging to the set $Z' \subset \mathcal{Z}$ is first reached, given that play in the ε-perturbed model begins in state $z \in \mathcal{Z}$. We deal with the question of how quickly a system converges to equilibria (limit sets) by characterizing the behavior of $W_x^*(z', N) = \max_{z \in \mathcal{Z}} W(z, z', N)$ for each $z' \in \Omega_x$ for each $x = m, f$. The introduction of the notion of basin of attraction of a limit set is useful for the analysis. The basin of attraction of $\omega \in \Omega_x$, denoted by $D(\omega)$, is the set of initial states from which the unperturbed process converges to ω with probability one, i.e.,

$$D(\omega) = \{z \in \mathcal{Z} | \exists T \ s.t. \ p_x^\tau(z, \omega, 0) = 1 \ \forall \tau > T\}.$$

Theorem 2. *For a sufficiently small ε, we have*

$$\begin{aligned}
W_m^*(\{\vec{B}\}, N) &= O(N(1-r)^{-\frac{(1-q^*)N}{2r}}),\\
W_m^*(\{\vec{A}\}, N) &= O(N(1-r)^{-\frac{q^*N}{2r}}),\\
W_f^*(\{\vec{B}\}, N) &= O(N),\\
W_f^*(\{\vec{A}\}, N) &= O(N(1-r)^{-\frac{N}{2r}}),
\end{aligned}$$

as $N \to \infty$[3].

Proof. See Appendix D □

A couple of remarks are in order. First, the ratio $\frac{W_m^*(\{\vec{B}\},N)}{W_m^*(\{\vec{A}\},N)}$ of waiting times can become arbitrarily large if the population size is large since $q^* < \frac{1}{2}$. The ratio $\frac{W_f^*(\{\vec{A}\},N)}{W_f^*(\{\vec{B}\},N)}$ of waiting times also increases in the size. This implies that $\vec{A}$ is more persistent than $\vec{B}$ in $G_m(N+1)$ and that $\vec{B}$ is, on the other hand, more persistent than $\vec{A}$ in $G_f(N+1)$ in the case of large population. Second, $W_f^*(\{\vec{B}\}, N)$, $W_m^*(\{\vec{B}\}, N)$, and $W_m^*(\{\vec{A}\}, N)$ have two effects on delaying convergence in the case of large population. One effect is that the waiting times until the number of players playing B expands to include the entire population are of the order of N. The other effect, significantly longer, is that the waiting times until player 0 does not have an opportunity for strategy revision for the periods required to reach the basin of attraction of the equilibrium increases exponentially in the size. However, the latter effect is eliminated in $W_f^*(\{\vec{B}\}, N)$ because in $G_f(N+1)$, the state wherein player 0 and two adjacent players from $\mathcal{I}_c$ (independent of N) play B are in the basin of attraction of $\vec{B}$, which only requires the waiting time that is independent of the population size. Although the waiting time for reaching $\vec{B}$ in $G_f(N+1)$, as expected, approaches infinity as $N \to \infty$, simulation results presented in the next subsection show that the expected waiting time for $\vec{B}$ in $G_f(N+1)$ is reasonably fast even in the case of an extremely large N.

4.2 Simulation Results

Table 1 shows the simulation results of the expected waiting times – $W_m(\vec{A}, \mathcal{B}, N)$, $W_m(\vec{B}, \mathcal{A}, N)$, $W_f(\vec{A}, \mathcal{B}, N)$, and $W_f(\vec{B}, \mathcal{A}, N)$, where $\mathcal{A} \subset \mathcal{Z}$ is the set of states wherein more than 95 percent players play A and $\mathcal{B} \subset \mathcal{Z}$ is the set of states wherein more than 95 percent players play B, with $r = 0.5$, $\varepsilon = 0.05$, $q^* = 0.04$, and N varying between 10 and 50[4]. It is evident that the ratios $\frac{W_m(\vec{A},\mathcal{B},N)}{W_m(\vec{B},\mathcal{A},N)}$ and $\frac{W_f(\vec{B},\mathcal{A},N)}{W_f(\vec{A},\mathcal{B},N)}$

[3] We use $f(N) = O(g(N))$ as $N \to \infty$ if there exists a constant C and $\bar{N}$ such that $|f(N)| \leqslant C|g(N)|$ for all $N \geqslant \bar{N}$.

[4] Simulation results in this section do not change significantly if $\mathcal{A}$ ($\mathcal{B}$) is replaced by the set of states where more than 95 percent of players *including player* 0 play A (B).

Comparison of waiting times

Expected Waiting Time

	$N=10$	$N=20$	$N=30$	$N=50$
$W_m(\vec{A},\mathcal{B},N)$	963	1717	2699	4781
$W_m(\vec{B},\mathcal{A},N)$	580	701	898	1083
$W_f(\vec{A},\mathcal{B},N)$	21	23	23	25
$W_f(\vec{B},\mathcal{A},N)$	7319	34765	127325	>130000

Table 1: Comparison of waiting times $W_m(\vec{A},\mathcal{B},N)$, $W_m(\vec{B},\mathcal{A},N)$, $W_f(\vec{A},\mathcal{B},N)$, and $W_f(\vec{B},\mathcal{A},N)$ with $r=0.5$, $\varepsilon=0.05$, $q^*=0.04$, and N varying between 10 and 50. $\mathcal{A}\subset\mathcal{Z}$ ($\mathcal{B}\subset\mathcal{Z}$) is the set of states wherein more than 95 percent of players play A (B).

Waiting times $W_f(\vec{A},\mathcal{B},N)$

population size	10	10^2	10^3	10^4
$W_f(\vec{A},\mathcal{B},N)$	21	25	32	37

Table 2: Comparison of waiting times $W_f(\vec{A},\mathcal{B},N)$, with $r=0.5$, $\varepsilon=0.05$, and N varying between 10 and 10^4.

increase as N increases. This implies that $\vec{A}$ is more persistent than $\vec{B}$ in $G_m(N+1)$ while $\vec{B}$ is more persistent than $\vec{A}$ in $G_f(N+1)$ in the case of a large population.

Table 2 shows the simulation result of the expected waiting times $W_f(\vec{A},\mathcal{B},N)$, where $r=0.5$, $\varepsilon=0.05$, and the population size varies. Although the expected waiting time for attaining $\mathcal{B}$ increases with the population size, it is reasonably fast even in the case of an extremely large N.

5 Emergence of Coexistence of Conventions as the Long-run Stochastically Stable State

The question of equilibria of local interaction games where different players take different actions has been studied as the question of coexistence of conventions.

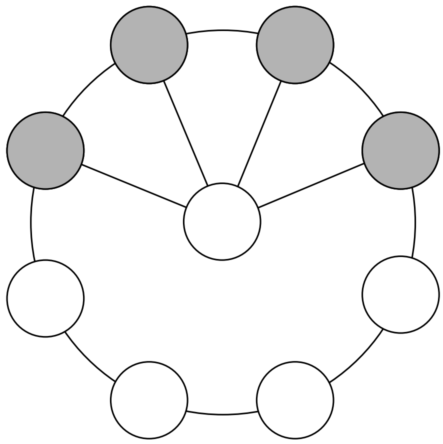

Figure 7: Network ($N = 8$, $M = 4$) – Only shaded players interact with player 0.

Berninghaus and Schwalbe (1996), Goyal and Janssen (1997), Morris (2000), and Sugden (1995) discussed the properties of the interaction structure that make the coexistence of conventions possible. This section provides an example of a network with one forward-looking player such that the state where conventions coexist would be long-run stochastically stable.

Let $\mathcal{I}_{c_1} = \{1, \ldots, M\} \subset \mathcal{I}$ and $\mathcal{I}_{c_2} = \{M+1, \ldots, N\} \subset \mathcal{I}$, where $3 \leqslant M \leqslant 4$. Players in $\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ are positioned around a circle, each $i \in \mathcal{I}_{c_1}$ interacts with his two immediate neighbors and player 0, and each $i \in \mathcal{I}_{c_2}$ interacts only with his two immediate neighbors. More formally,

$$\Phi(0) = \mathcal{I}_{c_1},$$
$$\Phi(i) = \{0, i-1, i+1\} \quad \forall i \in \mathcal{I}_{c_1},$$
$$\Phi(i) = \{i-1, i+1\} \quad \forall i \in \mathcal{I}_{c_2},$$

($i - 1 = N$ if $i = 1$, and $i + 1 = 1$ if $i = N$). Figure 7 shows the resulting network where $N = 8$ and $M = 4$. We consider two networks, $\tilde{G}_f(M, N+1)$, and $\tilde{G}_m(M, N+1)$ where in the case of $\tilde{G}_f(M, N+1)$, player 0 is forward-looking and players in $\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ are myopic, and in $\tilde{G}_m(M, N+1)$ all players behave myopically.

The behavior of the players may be described as follows. Each player in $\mathcal{I}_{c_1}$ plays A (B) if two or three of his neighbors played A (B) in the previous period. Each player in $\mathcal{I}_{c_2}$ plays A if at least one of his neighbors played A in the previous period because $q^* \in [\frac{1}{3}, \frac{1}{2})$. If player 0 is forward-looking, he plays B if at least one of his neighbors (players in $\mathcal{I}_{c_1}$) played B in the previous period. If player 0 is myopic, he plays A if $q_0 > q^*$ and B if $q_0 \leqslant q^*$[5].

Let $\vec{C}$ be the state wherein players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$ play B and players in $\mathcal{I}_{c_2}$ play A, and $\tilde{\Omega}_x$ ($x = m, f$) be the collection of all limit sets in $\tilde{G}_x(M, N+1)$. Proposition

[5]Recall that q_0 is the fraction of player 0's opponents who played A in the previous period.

3 computes the limit sets in these evolutionary system.

Proposition 3. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $3 \leqslant M \leqslant N-4$*. Then for each* $x = m, f$, $\tilde{\Omega}_x = \{\{\vec{A}\}, \{\vec{B}\}, \{\vec{C}\}\}$.

Proof. See Appendix E ☐

The third limit set $\vec{C}$ can be long-run stochastically stable in $\tilde{G}_f(M, N+1)$. This may be elaborated as follows. As seen in Theorem 1, efficient considerations tend to prevail in the region with the forward-looking player (players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$). On the other hand, as seen in Ellison's (1993) model, risk-dominant considerations tend to prevail in the region without the forward-looking player (players in $\mathcal{I}_{c_2}$). Thus, the system can easily move from state $\vec{A}$ or $\vec{B}$ to $\vec{C}$.

Theorem 3. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $3 \leqslant M \leqslant N-4$*. Then,*

1. *In* $\tilde{G}_m(M, N+1)$, $\vec{A}$ *is the unique long-run stochastically stable state.*

2. *In* $\tilde{G}_f(M, N+1)$, *both* $\vec{A}$ *and* $\vec{C}$ *are long-run stochastically stable states.*

Proof. See Appendix F. ☐

In the proof of Theorem 3, the cases of $\tilde{G}_m(M, N+1)$ and $\tilde{G}_f(M, N+1)$ defer with respect to $C(\vec{A}, \vec{C})$ ($C(\vec{A}, \vec{C}) > 1$ in $\tilde{G}_m(M, N+1)$; on the other hand, $C(\vec{A}, \vec{C}) = 1$ in $\tilde{G}_f(M, N+1)$). In $\tilde{G}_f(M, N+1)$, due to the emergence by patient play of the forward-looking player, the system can move from state $\vec{A}$ to $\vec{C}$ with minimum cost.

6 Conclusion

The local interaction model with one sufficiently patient forward-looking player was analyzed. Due to the emergence of patient play by the forward-looking player, efficient equilibrium is uniquely selected as the long-run stochastically stable state. Furthermore, in the case of a large population size, the risk-dominant equilibrium is more persistent than the Pareto-efficient equilibrium in the system without a forward-looking player. The Pareto-efficient equilibrium, on the other hand, is more persistent than the risk-dominant equilibrium in the system with a forward-looking player. Furthermore, if a cluster of myopic players does not interact with the forward-looking player, the coexistence of conventions can be the long-run stochastically stable state.

This model may be extended in many directions, for instance the model may be extended to deal with the network, the matching rule, the society with two or more forward-looking players, the incompleteness of knowledge for prediction of the future by the forward-looking player, and so on. These areas should be considered for future research.

Appendix

A Proof of Proposition 1

Proposition 1. *Let $q^* \in [\frac{1}{3}, \frac{1}{2})$ and $N \geqslant 3$.*

1. *(Without noise) There exists $\bar{\delta} \in (0,1)$ such that for all $\delta \in (\bar{\delta}, 1)$,*

$$\sigma^*(\hat{z}) = \begin{cases} A & if\ \hat{z} = (A, A, \dots, A) \text{ or } \hat{z} = (B, A, \dots, A) \\ B & \text{otherwise.} \end{cases}$$

2. *(With noise) Given that $\delta \in (\bar{\delta}, 1)$, there exists $\bar{\varepsilon}(\delta) > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}(\delta))$, the above strategy is still optimal.*

Proof. Let $\pi(\sigma, \varepsilon, \hat{z})$ denote the expected present discounted payoff in the current period with player 0 following the strategy σ under $\varepsilon \geqslant 0$ and the state in the previous period being $\hat{z}$. We first consider the case without noise. Let σ' be the strategy that player 0 deviates just once from σ^* in the current period. We will show that

$$\exists \bar{\delta} \in (0,1), \forall \delta \in (\bar{\delta}, 1), \forall \hat{z} \in \mathcal{Z}, \quad \pi(\sigma^*, 0, \hat{z}) > \pi(\sigma', 0, \hat{z}).$$

Let $v^\tau(z, z', \sigma)$ denote the probability that the combination of the myopic best responses of the players in $\mathcal{I}_c$ and the strategy σ of player 0 causes the system to move from the state z to z' in τ steps, and let $\bar{v}(\hat{z}, z, z') = \lim_{\tau \to \infty} v^\tau(\hat{z}, z, z')$. We proceed first by establishing the following two lemmas.

Lemma 1. *$\forall \hat{z} \in \mathcal{Z}$, for $\sigma = \sigma^*, \sigma'$,*

$$\bar{v}(\hat{z}, \vec{A}, \sigma) + \bar{v}(\hat{z}, \vec{B}, \sigma) = 1.$$

Proof. First consider σ^*. Suppose that state $\vec{A}$ is attained. In $\vec{A}$, each player in $\mathcal{I}_c$ with an opportunity to revise his strategy choice will not change his action because all his neighbors play A. Moreover, player 0 with the opportunity to revise his strategy choice will also not change his action because all his neighbors play A. Hence, once state $\vec{A}$ is attained, the process cannot be led out of $\vec{A}$. Similarly, once state $\vec{B}$ is reached, the process cannot be led out of $\vec{B}$. In order to show that all the states except $\vec{A}$ and $\vec{B}$ are unstable, we show that for all $z \in \mathcal{Z} \setminus \{\vec{A}, \vec{B}\}$, the system can attain either state $\vec{A}$ or $\vec{B}$ in a finite number of steps. First, consider the state where at least one player in $\mathcal{I}_c$ plays B. It suffices to show that the state wherein all players except player $i \in \mathcal{I}_c$ play A can reach state $\vec{B}$. Suppose that player 0 has an opportunity for strategy revision, while i does not. The system will then attain the state wherein players 0 and i play B in the next period. Furthermore, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while i does not. In that case, the system will attain the state wherein players 0, $i-1$, i, and $i+1$ play B in the next period. Furthermore, suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. In that case, $i-2$ and $i+2$ will change their actions from A to B. Since N is finite, by repeating this argument, it can be shown that the system will attain state $\vec{B}$. The remaining case is that pertaining to $(B, A, \ldots, A)$. With player 0 having an opportunity for strategy revision, state $\vec{A}$ is reached. Hence, for all $\hat{z} \in \mathcal{Z}$, $\bar{v}(\hat{z}, \vec{A}, \sigma^*) + \bar{v}(\hat{z}, \vec{B}, \sigma^*) = 1$.

Next, consider σ'. After player 0 deviates from σ^* in the current period, he will select the same action as σ^* when he has an opportunity for strategy revision. Thus, according to the abovementioned argument, the system eventually leads all states to $\vec{A}$ or $\vec{B}$. Hence, for all $\hat{z} \in \mathcal{Z}$, $\bar{v}(\hat{z}, \vec{A}, \sigma') + \bar{v}(\hat{z}, \vec{B}, \sigma') = 1$. □

Lemma 2. *$\forall \hat{z} \in \tilde{Z} \equiv \mathcal{Z} \setminus \{\vec{A}, \vec{B}, (A, B, \ldots, B), (B, A, \ldots, A)\}$,*

1. *$\exists \gamma_1 > 0$, $\exists \tau_1 > 0$, $\forall \tau > \tau_1$, $v^\tau(\hat{z}, \vec{B}, \sigma^*) - v^\tau(\hat{z}, \vec{B}, \sigma') > \gamma_1$,*
2. *$\forall \gamma_2 > 0$, $\exists \tau_2 > 0$, $\forall \tau > \tau_2$,*

$$|(v^\tau(\hat{z}, \vec{A}, \sigma^*) - v^\tau(\hat{z}, \vec{A}, \sigma')) + (v^\tau(\hat{z}, \vec{B}, \sigma^*) - v^\tau(\hat{z}, \vec{B}, \sigma'))| < \gamma_2.$$

Proof. (the proof of part 1) First, we show that for all $\hat{z} \in \tilde{Z}$, $\bar{v}(\hat{z}, \vec{B}, \sigma^*) > \bar{v}(\hat{z}, \vec{B}, \sigma')$[6]. If player 0 follows σ', he plays A in the current period when $\hat{z} \in \tilde{Z}$. Therefore, there exist cases such that player 0 will continue to play A for sometime because of unavailability of an opportunity for strategy revision. If player 0 continues to play A for sometime, there exist cases in which the process will lead $\hat{z} \in \tilde{Z}$ to $\vec{A}$, and if he has followed σ^*, the process could lead $\hat{z} \in \tilde{Z}$ to $\vec{B}$. In order to observe this, it suffices to show that with player 0 continuing to play A, the state where in all players except players 0 and $i \in \mathcal{I}_c$ play B can reach state $\vec{A}$ in a finite number of steps. Suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while i does not. The system will then attain the state wherein players 0, $i-1$, i, and $i+1$ play A in the next period. Furthermore, suppose that $i-2$ and $i+2$ have opportunities for strategy revision. In that case, players $i-2$ and $i+2$ will change their actions from B to A. Since N is finite, by repeating this argument, it can be shown that the system will attain state $\vec{A}$. Hence, $\bar{v}(\hat{z}, \vec{B}, \sigma^*) > \bar{v}(\hat{z}, \vec{B}, \sigma')$. Let $\gamma_4 = \bar{v}(\hat{z}, \vec{B}, \sigma^*) - \bar{v}(\hat{z}, \vec{B}, \sigma') > 0$. Further, let $\gamma_1 \in (0, \gamma_4)$. Since the sequences $\{v^\tau(\hat{z}, \vec{B}, \sigma^*)\}_{\tau=1}^\infty$ and $\{v^\tau(\hat{z}, \vec{B}, \sigma')\}_{\tau=1}^\infty$ converge to $\bar{v}(\hat{z}, \vec{B}, \sigma^*)$ and $\bar{v}(\hat{z}, \vec{B}, \sigma')$, respectively, there exists $\tau_1 > 0$ such that for all $\tau > \tau_1$,

[6] Once state $\vec{B}$ is reached, the process cannot be led out of $\vec{B}$. This implies that the sequences $\{v^\tau(\hat{z}, \vec{B}, \sigma^*)\}_{\tau=1}^\infty$ and $\{v^\tau(\hat{z}, \vec{B}, \sigma')\}_{\tau=1}^\infty$ are monotone increasing. Furthermore, since $v^\tau(\hat{z}, \vec{B}, \cdot)$ is a probability, $\{v^\tau(\hat{z}, \vec{B}, \sigma^*)\}_{\tau=1}^\infty$ and $\{v^\tau(\hat{z}, \vec{B}, \sigma')\}_{\tau=1}^\infty$ are bounded. Hence, $\{v^\tau(\hat{z}, \vec{B}, \sigma^*)\}_{\tau=1}^\infty$ and $\{v^\tau(\hat{z}, \vec{B}, \sigma')\}_{\tau=1}^\infty$ have limits.

$|v^\tau(\hat{z},\vec{B},\sigma^*) - \bar{v}(\hat{z},\vec{B},\sigma^*)| < \frac{\gamma_4-\gamma_1}{2}$ and $|v^\tau(\hat{z},\vec{B},\sigma') - \bar{v}(\hat{z},\vec{B},\sigma')| < \frac{\gamma_4-\gamma_1}{2}$. Then, we have

$$\begin{aligned} v^\tau(\hat{z},\vec{B},\sigma^*) - v^\tau(\hat{z},\vec{B},\sigma') &> \left[\bar{v}(\hat{z},\vec{B},\sigma^*) - \frac{\gamma_4-\gamma_1}{2}\right] - \left[\frac{\gamma_4-\gamma_1}{2} + \bar{v}(\hat{z},\vec{B},\sigma')\right] \\ &= \bar{v}(\hat{z},\vec{B},\sigma^*) - \bar{v}(\hat{z},\vec{B},\sigma') - \gamma_4 + \gamma_1 \\ &= \gamma_1. \end{aligned}$$

(the proof of part 2) By Lemma 1, the sequence $\{(v^\tau(\hat{z},\vec{A},\sigma^*) + v^\tau(\hat{z},\vec{B},\sigma^*)) - (v^\tau(\hat{z},\vec{A},\sigma') + v^\tau(\hat{z},\vec{B},\sigma'))\}_{\tau=1}^\infty$ converges to 0. Thus, for any $\gamma_2 > 0$, there exists $\tau_2 > 0$ such that for all $\tau > \tau_2$,

$$|(v^\tau(\hat{z},\vec{A},\sigma^*) + v^\tau(\hat{z},\vec{B},\sigma^*)) - (v^\tau(\hat{z},\vec{A},\sigma') + v^\tau(\hat{z},\vec{B},\sigma'))| < \gamma_2.$$

□

B Proof of Proposition 2

Proposition 2. *Let $q^* \in [\frac{1}{3},\frac{1}{2})$ and $N \geqslant 3$. Then for each $x = m, f$, $\Omega_x = \{\{\vec{A}\},\{\vec{B}\}\}$.*

Proof. First, consider $G_m(N+1)$. In $\vec{A}$, each player with an opportunity to revise his strategy choice will not change his action because all his neighbors play A. Hence, $\vec{A}$ is a limit set. Similarly, $\vec{B}$ is a limit set. In order to demonstrate that there exist no other limit sets, we show that for all $z \in \mathcal{Z} \setminus \{\vec{A},\vec{B}\}$, the system can attain either state $\vec{A}$ or $\vec{B}$ in a finite number of steps. Consider the state where among the players in $\mathcal{I}_c$, players who play A and B co-exist. It suffices to show that the state wherein player 0 and one player in $\mathcal{I}_c$ play B can reach state $\vec{B}$ and the state wherein player 0 and one player in $\mathcal{I}_c$ play A can reach state $\vec{A}$. Consider the state wherein all players except player 0 and player $i \in \mathcal{I}_c$ play A (B). Suppose that player 0 will not have an opportunity for strategy revision for sometime. Further, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while i does not. Then, the system will reach the state wherein players 0, $i-1$, i, and $i+1$ play B (A) in the next period. Furthermore, suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. In that case, players $i-2$ and $i+2$ will change their actions from A (B) to B (A). Since N is finite, by repeating this argument, it can be shown that the system will reach state $\vec{B}$ ($\vec{A}$). The remaining cases pertain to states $(B, A, \ldots, A)$ and $(A, B, \ldots, B)$. With player 0 having an opportunity for strategy revision, these states reach $\vec{A}$ and $\vec{B}$, respectively. Hence, $\Omega_m = \{\{\vec{A}\},\{\vec{B}\}\}$.

Next, we consider $G_f(N+1)$. The proof for this case is similar to the proof of Lemma 1. In $\vec{A}$, each player in $\mathcal{I}_c$ with an opportunity to revise his choice will not change his action because all his neighbors play A. Moreover, player 0 with an opportunity to revise his choice will also not change his action because all his neighbors play A. Hence, $\vec{A}$ is a limit set. Similarly, $\vec{B}$ is a limit set. In order to demonstrate that there are no other limit sets, we show that for all $z \in \mathcal{Z} \setminus \{\vec{A}, \vec{B}\}$, the system can attain either state $\vec{A}$ or $\vec{B}$ in a finite number of steps. First, consider the state wherein at least one player in $\mathcal{I}_c$ plays B. It suffices to show that the state wherein all players except player $i \in \mathcal{I}_c$ play A can reach state $\vec{B}$. Suppose that player 0 has an opportunity for strategy revision while i does not. Then, the system will reach the state wherein players 0 and i play B in the next period. Furthermore, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision while i does not. Then, the system will reach the state wherein players 0, $i-1$, i, and $i+1$ play B in the next period. Furthermore, suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. In that case, players $i-2$ and $i+2$ will change their actions from A to B. Since N is finite, by repeating this argument, it can be shown that the system will attain state $\vec{B}$. The remaining case is that pertaining to state $(B, A, \ldots, A)$. With player 0 having an opportunity for strategy revision, this state reaches state $\vec{A}$. Hence, $\Omega_f = \{\{\vec{A}\}, \{\vec{B}\}\}$. □

Let $\Delta(\hat{z}) = \pi(\sigma^*, 0, \hat{z}) - \pi(\sigma', 0, \hat{z})$ and let ρ be the probability that player 0 is matched with some players (note that clearly, $\rho > 0$). When $\hat{z} = (A, A, \ldots, A)$ or $\hat{z} = (B, A, \ldots, A)$ we have

$$\Delta(\hat{z}) = \sum_{s=0}^{\infty} (1-r)^s \delta^s \rho(a-c) > 0.$$

Similarly, when $\hat{z} = (B, B, \ldots, B)$ or $\hat{z} = (A, B, \ldots, B)$, we have

$$\Delta(\hat{z}) = \sum_{s=0}^{\infty} (1-r)^s \delta^s \rho(b-d) > 0.$$

Consider the case wherein $\hat{z} \in \tilde{Z}$. Let $\tau' = \max\{\tau_1, \tau_2\}$, γ_2 satisfies $b > a + \frac{a}{\gamma_1}\gamma_2$, and $\gamma_3 = b - a - \frac{a}{\gamma 1}\gamma_2 > 0$. Since

$$\pi(\sigma, 0, \hat{z}) = \sum_{z \in \mathcal{Z}} \sum_{s=0}^{\infty} v^s(\hat{z}, z, \sigma) \delta^s u(z, \sigma),$$

where $u(z, \sigma)$ is the expected payoff of the stage game from following σ in z, we obtain, using Lemma 2,

$$
\begin{aligned}
\Delta(\hat{z}) &= \sum_{s=\tau'+1}^{\infty}[v^s(\hat{z},\vec{A},\sigma^*)-v^s(\hat{z},\vec{A},\sigma')]\delta^s\rho a + \\
&\quad + \sum_{s=\tau'+1}^{\infty}[v^s(\hat{z},\vec{B},\sigma^*)-v^s(\hat{z},\vec{B},\sigma')]\delta^s\rho b + K(\delta) \\
&> \sum_{s=\tau'+1}^{\infty}[-v^s(\hat{z},\vec{B},\sigma^*)+v^s(\hat{z},\vec{B},\sigma')-\gamma_2]\delta^s\rho a + \\
&\quad + \sum_{s=\tau'+1}^{\infty}[v^s(\hat{z},\vec{B},\sigma^*)-v^s(\hat{z},\vec{B},\sigma')]\delta^s\rho\left(a+\frac{a}{\gamma_1}\gamma_2+\gamma_3\right) + K(\delta) \\
&> \sum_{s=\tau'+1}^{\infty}\delta^s\rho\gamma_1\gamma_3 + K(\delta),
\end{aligned}
$$

where

$$
\begin{aligned}
K(\delta) &= \sum_{s=0}^{\tau'}[v^s(\hat{z},\vec{A},\sigma^*)-v^s(\hat{z},\vec{A},\sigma')]\delta^s\rho a + \sum_{s=0}^{\tau'}[v^s(\hat{z},\vec{B},\sigma^*)-v^s(\hat{z},\vec{B},\sigma')]\delta^s\rho b \\
&\quad + \sum_{z\in Z\setminus\{\vec{A},\vec{B}\}}\sum_{s=0}^{\infty}v^s(\hat{z},z,\sigma^*)\delta^s u(z,\sigma^*) - \sum_{z\in Z\setminus\{\vec{A},\vec{B}\}}\sum_{s=0}^{\infty}v^s(\hat{z},z,\sigma')\delta^s u(z,\sigma').
\end{aligned}
$$

Since $\rho, \gamma_1, \gamma_3 > 0$, $\sum_{s=\tau'+1}^{\infty}\delta^s\rho\gamma_1\gamma_3 \to +\infty$ as $\delta \to 1$. Since for all $z \in \mathcal{Z} \setminus \{\vec{A}, \vec{B}\}$, $v^\tau(\hat{z}, z, \sigma^*) \to 0$ and $v^\tau(\hat{z}, z, \sigma') \to 0$ as $\tau \to \infty$ by Lemma 1, $K(\delta)$ has a finite limit as $\delta \to 1$. Hence, for all $\hat{z} \in \mathcal{Z}$, there exists $\delta(\hat{z}) \in (0,1)$ such that for all $\delta \in (\delta(\hat{z}), 1)$ $\Delta(\hat{z}) > 0$. Assigning $\bar{\delta} = \max_{\hat{z}\in\mathcal{Z}} \delta(\hat{z})$ ($\bar{\delta} \in (0,1)$ since $|\mathcal{Z}| < \infty$) completes the proof.

Next, consider the case with noise wherein δ is given as $\delta \in (\bar{\delta}, 1)$. Let $\tilde{\Delta} = \min_{\hat{z}\in\mathcal{Z}}[\pi(\sigma^*, 0, \hat{z}) - \pi(\sigma', 0, \hat{z})]$. Since $|\mathcal{Z}| < \infty$, $\tilde{\Delta} > 0$. Since $\pi(\sigma, \varepsilon, \hat{z})$ is a continuous function in ε, it is evident that $\lim_{\varepsilon\to 0} \pi(\sigma, \varepsilon, \hat{z}) = \pi(\sigma, 0, \hat{z})$. Therefore, there exists $\bar{\varepsilon}(\delta) > 0$ such that for all $\varepsilon < \bar{\varepsilon}(\delta)$, for all $\hat{z} \in \mathcal{Z}$, $|\pi(\sigma^*, 0, \hat{z}) - \pi(\sigma^*, \varepsilon, \hat{z})| < \frac{\tilde{\Delta}}{2}$ and $|\pi(\sigma', 0, \hat{z}) - \pi(\sigma', \varepsilon, \hat{z})| < \frac{\tilde{\Delta}}{2}$. Thus, we have

$$
\begin{aligned}
\pi(\sigma^*,\varepsilon,\hat{z}) - \pi(\sigma',\varepsilon,\hat{z}) &> \left[\pi(\sigma^*,0,\hat{z}) - \frac{\tilde{\Delta}}{2}\right] - \left[\frac{\tilde{\Delta}}{2} + \pi(\sigma',0,\hat{z})\right] \\
&= \pi(\sigma^*,0,\hat{z}) - \pi(\sigma',0,\hat{z}) - \tilde{\Delta} \\
&\geqslant 0.
\end{aligned}
$$

This implies that one deviation property holds when $\varepsilon \in (0, \bar{\varepsilon}(\delta))$. □

C Proof of Theorem 1

Theorem 1. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $N \geqslant 3$. *Then,* *1. In $G_m(N+1)$, both $\vec{A}$ and $\vec{B}$ are the long-run stochastically stable states.*

2. In $G_f(N+1)$, $\vec{B}$ is the unique long-run stochastically stable state.

Proof. (proof for the first part) We aim to show that $C(\vec{A},\vec{B}) = C(\vec{B},\vec{A}) = 2$. We first show that $C(\vec{A},\vec{B}), C(\vec{B},\vec{A}) \neq 1$. Suppose that in $\vec{A}$ ($\vec{B}$), player $i \in \mathcal{I}_c$ mutates and changes his action from A (B) to B (A). Each player $j \in \mathcal{I}_c \setminus \{i\}$ with an opportunity for strategy revision will not change his action because each player has two or three neighbors who play A (B). Player 0 with an opportunity for strategy revision will not change his action because $N-1$ of his neighbors play A (B) (we consider the case wherein $N \geqslant 3$). With player i having an opportunity for strategy revision, the system will revert to $\vec{A}$ ($\vec{B}$). Suppose that in $\vec{A}$ ($\vec{B}$), player 0 mutates and changes his action from A (B) to B (A). Each player in $\mathcal{I}_c$ with an opportunity to revise his choice will play A (B) because he has two neighbors who play A (B). With player 0 having an opportunity for strategy revision, the system will again revert to $\vec{A}$ ($\vec{B}$). Thus, there exist no paths such that a single mutation and the unperturbed process can change the state of the system from $\vec{A}$ ($\vec{B}$) to $\vec{B}$ ($\vec{A}$). Hence, $C(\vec{A},\vec{B}), C(\vec{B},\vec{A}) \neq 1$.

Suppose that in $\vec{A}(\vec{B})$, players 0 and $i \in \mathcal{I}_c$ mutate and change their actions from A (B) to B (A). Furthermore, suppose that player 0 will not have an opportunity for strategy revision for sometime. Also, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while player i does not. Then, the system will reach a state wherein players 0, $i-1$, i, and $i+1$ play B (A) in the next period. Suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. In that case, players $i-2$ and $i+2$ will change their actions from B (A) to A (B). Since N is finite, the system will reach the state $\vec{B}$ ($\vec{A}$) by repeating this argument. Hence, $C(\vec{A},\vec{B}) = C(\vec{B},\vec{A}) = 2$.

(proof for the second part) We aim to show that $C(\vec{A},\vec{B}) = 1$ and $C(\vec{B},\vec{A}) > 1$. We first consider $C(\vec{A},\vec{B})$. Suppose that in $\vec{A}$, player $i \in \mathcal{I}_c$ mutates and changes his action from A to B. Also, suppose that player 0 has an opportunity for strategy revision, while player i does not. Then, the system will reach a state wherein players 0 and i play B in the next period. Furthermore, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while player i does not. Then, the system reaches a state wherein players 0, $i-1$, i, and $i+1$ play B in the next period. Furthermore, suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. In that case, players $i-2$ and $i+2$ will change their actions from A to B. Since N is finite, the system will reach the state $\vec{B}$ by repeating this argument. Hence, $C(\vec{A},\vec{B}) = 1$.

Next, we consider $C(\vec{B},\vec{A})$. It suffices to show that $C(\vec{B},\vec{A}) \neq 1$. Suppose that in $\vec{B}$, player $i \in \mathcal{I}_c$ mutates and changes his action from B to A. Each player $j \in \mathcal{I}_c \setminus \{i\}$ with an opportunity for strategy revision will not change his action because each player has two or three neighbors who play B. Player 0 with an opportunity for strategy revision will not change his action because $N-1$ of his neighbors play B. With player i having an opportunity for strategy revision, the system will again revert to $\vec{B}$. Suppose that in $\vec{B}$, player 0 mutates and changes his action from B to A. Each player in $\mathcal{I}_c$ with the opportunity to revise his choice will play B because he has two neighbors who play B. With player 0 having an opportunity for strategy revision, the system will again revert to $\vec{B}$. Thus, there exist no paths such that a single mutation and the unperturbed process can change the state of the system from $\vec{B}$ to $\vec{A}$. Hence, $C(\vec{B},\vec{A}) > 1$[7] by repeating this argument. □

D Proof of Theorem 2

Theorem 2. *For a sufficiently small ε, we have*

$$\begin{aligned}
&W_m^*(\{\vec{B}\},N) = O(N(1-r)^{-\frac{(1-q^*)N}{2r}}),\\
&W_m^*(\{\vec{A}\},N) = O(N(1-r)^{-\frac{q^*N}{2r}}),\\
&W_f^*(\{\vec{B}\},N) = O(N),\\
&W_f^*(\{\vec{A}\},N) = O(N(1-r)^{-\frac{N}{2r}}),
\end{aligned}$$

as $N \to \infty$.

Proof. Consider $W_m^*(\{\vec{B}\},N)$. The system leads to either $\vec{A}$ or $\vec{B}$ from any initial state after some waiting time of the order $C_1 N$ (C_1 is a constant). Therefore, we have

$$W_m(z,\{\vec{B}\},N) \leqslant C_1 N + W_m(\vec{A},\{\vec{B}\},N) \quad \forall z \in \mathcal{Z}. \tag{1}$$

We focus on the paths from $\vec{A}$ to $\vec{B}$ that involve minimum costs (two mutations). Although there exist other paths that involve costs greater than minimum costs, the probability of occurrence of these paths occur is negligible because ε is sufficiently small. Player 0 and one player from $\mathcal{I}_c$ mutate to play B after a waiting time of at most $\frac{C_2}{N\varepsilon^2}$ (C_2 is a constant). Let $\Theta \subset \mathcal{Z}$ be the set of states wherein player 0 and one player from $\mathcal{I}_c$ play B and θ be a member of Θ, wherein player i plays B. Then, we have

$$W_m(\vec{A},\{\vec{B}\},N) \leqslant \frac{C_2}{N\varepsilon^2} + W_m(\theta,\{\vec{B}\},N), \tag{2}$$

which is derived from the symmetry:

$$W_m(\theta,\{\vec{B}\},N) = W_m(z,\{\vec{B}\},N) \quad \forall z \in \Theta.$$

[7] More accurately, $C(\vec{B},\vec{A}) = 2$. Suppose that in $\vec{B}$, player 0 and player $i \in \mathcal{I}_c$ mutate and change their actions from B to A. Furthermore, suppose that player 0 will not have an opportunity for strategy revision for sometime. Also, suppose that players $i-1$ and $i+1$ have opportunities for strategy revision, while i does not. Then, the system will reach a state wherein player 0, $i-1$, i and $i+1$ play A in the next period. Furthermore, suppose that players $i-2$ and $i+2$ have opportunities for strategy revision. They will then change their actions from B to A. The system will attain $\vec{A}$.

It can be readily observed that in $G_m(N+1)$, the states wherein player 0 and $(1-q^*)N$ adjacent players from $\mathcal{I}_c$ play B (denoted by $\tilde{z}$) are contained in $D(\{\vec{B}\})$. The system moves from the state θ to $\tilde{z}$ due to the occurrence of the event that players playing B expand from both sides of player i, while players 0 and i do not have opportunity for strategy revision (we refer to this event as E_1). Given that players 0 and i continue to play B, it takes a waiting time of $\frac{1}{2r}$ periods until players $i+1$ or $i-1$ play B and $\frac{(1-q^*)N}{2r}$ periods until $(1-q^*)N$ adjacent players from $\mathcal{I}_c$ play B. Thus, the probability that event E_1 occurs is at least $\psi = (1-r)^{\frac{1}{2r}+\frac{(1-q^*)N}{2r}}$. If E_1 occurs, the system will move from state θ to $\vec{B}$ after a waiting time of $\frac{N}{2r}$ periods. Therefore, we have

$$W_m(\theta, \{\vec{B}\}, N) \leqslant \psi \cdot \frac{N}{2r} + (1-\psi)\left(\frac{N}{2r} + W_m^*(\{\vec{B}\}, N)\right). \tag{3}$$

By (1), (2), and (3) we have

$$\begin{aligned} W_m(z, \{\vec{B}\}, N) \quad &\leqslant \quad C_1 N + \frac{C_2}{N\varepsilon^2} + \psi \cdot \frac{N}{2r} + \\ &\qquad +(1-\psi)\left(\frac{N}{2r} + W_m^*(\{\vec{B}\}, N)\right) \quad \forall z \in \mathcal{Z}. \end{aligned}$$

Considering the maximum value of the left-hand side over $z \in \mathcal{Z}$ yields

$$W_m^*(\{\vec{B}\}, N) \leqslant [C_3 N + C_4 N^{-1}](1-r)^{-\frac{(1-q^*)N}{2r}},$$

where C_3 and C_4 are constants, as desired.

The corresponding proofs for $W_m^*(\{\vec{A}\}, N)$, $W_f^*(\{\vec{B}\}, N)$, $W_f^*(\{\vec{A}\}, N)$ are analogous to the abovementioned argument. The only observation required is that in $G_m(N+1)$, the states wherein player 0 and (q^*N+1) adjacent players from $\mathcal{I}_c$ play A are contained in $D(\{\vec{A}\})$, while in $G_f(N+1)$, the states wherein player 0 and two adjacent players from $\mathcal{I}_c$ play B are contained in $D(\{\vec{B}\})$ and $D(\{\vec{A}\}) = \{\vec{A}, (B, A, \ldots, A)\}$. □

E Proof of Proposition 3

Proposition 3. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $3 \leqslant M \leqslant N-4$. *Then, for each* $x = m, f$, $\tilde{\Omega}_x = \{\{\vec{A}\}, \{\vec{B}\}, \{\vec{C}\}\}$.

Proof. Consider $\tilde{G}_m(M, N+1)$. In $\vec{A}$, each player with an opportunity for strategy revision will not change his action because all his neighbors play A. Hence, $\vec{A}$ is a limit set. Similarly, $\vec{B}$ is a limit set. In $\vec{C}$, players 1 and M will not change their actions because they have two neighbors who play B. players $M+1$ and N will not change their actions because they have a neighbor who play A. Player 0 and players in $\mathcal{I}_{c_1} \setminus \{1, M\}$ will not change their actions because all their neighbors play

B. Players $M+2$ through $N-1$ will not change their actions because all their neighbors play A. Hence, $\vec{C}$ is a limit set. In order to demonstrate that there exist no other limit sets, we show that for all $z \in \mathcal{Z} \setminus \{\vec{A}, \vec{B}, \vec{C}\}$, the system can reach state $\vec{A}$, $\vec{B}$, or $\vec{C}$ in a finite number of steps. First, we show that the state wherein player 0 and one player from $\mathcal{I}_{c_1}$ play A (B) can reach the state wherein all players in $\mathcal{I}_c \setminus \mathcal{I}_{c_2}$ play A (B). Consider the state wherein all players except player 0 and player $i \in \mathcal{I}_{c_1}$ play A (B). Suppose that player 0 will not have an opportunity for strategy revision for sometime. Also, suppose that i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision while i does not. Then, i's neighbors will change their actions from A (B) to B (A) in the next period. Furthermore, suppose that the neighbors of i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision. In that case, they will change their actions from A (B) to B (A). Since M is finite, by repeating this argument, it can be shown that the system will reach the state wherein all players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$ play B (A). Next, we show that the state wherein one player in $\mathcal{I}_{c_2}$ plays A can reach the state wherein all players in $\mathcal{I}_{c_2}$ play A. Consider the state wherein all players except player $i \in \mathcal{I}_{c_2}$ play B. Suppose that i's neighbors in $\mathcal{I}_{c_2}$ have opportunities for the strategy revision while i does not. In that case, i's neighbors will change their actions from B to A in the next period. Furthermore, suppose that the neighbors of i's neighbors in $\mathcal{I}_{c_2}$ have opportunities for strategy revision. They will then change their actions from B to A in the next period. By repeating this argument, it can be shown that the system can reach the state wherein all players in $\mathcal{I}_{c_2}$ play A. The remaining cases are that pertaining to states $(B, A, \ldots, A)$, $(A, B, \ldots, B)$, and the state wherein player 0 and the players in $\mathcal{I}_{c_2}$ play A and the players in $\mathcal{I}_{c_1}1$ play B. With player 0 having an opportunity for strategy revision, these states can reach states $\vec{A}$, $\vec{B}$, and $\vec{C}$. Hence, $\tilde{\Omega}_m = \{\{\vec{A}\}, \{\vec{B}\}, \{\vec{C}\}\}$.

Next, consider $\tilde{G}_f(M, N+1)$. In $\vec{A}$, each player in $\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ with an opportunity for strategy revision will not change his action because all his neighbors play A. Hence, $\vec{A}$ is a limit set. Similarly, $\vec{B}$ is a limit set. In $\vec{C}$, players 1 and M will not change their actions because they have two neighbors who play B. Players $M+1$ and N will not change their actions because they have a neighbor who plays A. Players in $\mathcal{I}_{c_1} \setminus \{1, M\}$ will not change their actions because all their neighbors play B. Players in $\mathcal{I}_{c_2} \setminus \{M+1, N\}$ will not change their actions because all their neighbors play A. In addition, player 0 with an opportunity for strategy revision will not change his action because all his neighbors play B. Hence, $\vec{C}$ is a limit set. In order to demonstrate that there are no other limit sets, we show that for all $z \in \mathcal{Z} \setminus \{\vec{A}, \vec{B}, \vec{C}\}$, the system can attain states $\vec{A}$, $\vec{B}$, or $\vec{C}$ in a finite number of steps. First, we show that the state wherein one player in $\mathcal{I}_{c_1}$ plays B can reach the state wherein all players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$ play B. Consider the state wherein all players except player $i \in \mathcal{I}_{c_1}$ play A. Suppose that player 0 has an opportunity for strategy revision while i does not. In that case, player 0 will change his action from A to B in the next period. Furthermore, suppose that i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision while i does not. Then, i's neighbors will change their actions from A to B in the next period. Furthermore, suppose that the neighbors of i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision. Then, they will change their actions from A to B. Since M is finite, by repeating this argument, it can be shown that the system will attain the state wherein all players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$ play B. By the same argument for the case of $\tilde{G}_m(M, N+1)$, we can observe that the state wherein one player in $\mathcal{I}_{c_2}$ plays A can reach the state wherein all players in $\mathcal{I}_{c_2}$ play A. The remaining case is that pertaining to state $(B, A, \ldots, A)$. With player 0 having an opportunity for strategy revision, this state can reach state $\vec{A}$. Hence, $\tilde{\Omega}_f = \{\{\vec{A}\}, \{\vec{B}\}, \{\vec{C}\}\}$. □

F Proof of Theorem 3

Theorem 3. *Let* $q^* \in [\frac{1}{3}, \frac{1}{2})$ *and* $3 \leqslant M \leqslant N-4$. *Then,*

1. *In* $\tilde{G}_m(M, N+1)$, $\vec{A}$ *is the unique long-run stochastically stable state.*

2. *In* $\tilde{G}_f(M, N+1)$, *both* $\vec{A}$ *and* $\vec{C}$ *are long-run stochastically stable states.*

Proof. (proof for the first part) First, we show that $C(\vec{B}, \vec{C})$, $C(\vec{C}, \vec{A}) = 1$ and $C(\vec{A}, \vec{B})$, $C(\vec{A}, \vec{C})$, $C(\vec{B}, \vec{A})$, $C(\vec{C}, \vec{B}) > 1$.

$C(\vec{B}, \vec{C}) = 1$: Suppose that in $\vec{B}$, player $i \in \mathcal{I}_{c_2} \cup \{1, M\}$ mutates and changes his action from B to A. Further, suppose that player i's neighbors in $\mathcal{I}_{c_2}$ have opportunities for strategy revision ,while player i does not. The system will then reach a state wherein i and his neighbors play A in the next period. Furthermore, the neighbors of i's neighbors who are in $\mathcal{I}_{c_2}$ have opportunities for strategy revision. They will then change their actions from B to A in the next period. By repeating this argument (and if a mutant is 1 or M, with the mutant having opportunity for strategy revision [8]), the system will attain state $\vec{C}$. Hence, $C(\vec{B}, \vec{C}) = 1$.

$C(\vec{C}, \vec{A}) = 1$: Suppose that in $\vec{C}$, player 0 mutates and changes his action from B to A and that he will not have an opportunity for strategy revision for sometime. Further, suppose that players 1 and M have opportunities for strategy revision. Then, they will change their actions from B to A in the next period. Furthermore, players 2 and $M-1$ with opportunities for strategy revision will change their actions from B to A. The system will reach state $\vec{A}$ by repeating this argument. Hence, $C(\vec{C}, \vec{A}) = 1$.

$C(\vec{A}, \vec{B}) > 1$: Suppose that in $\vec{A}$, player 0 mutates and changes his action from A to B. Each player $i \in \mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ with an opportunity for strategy revision will not change his action because each player has two neighbors who play A. With player 0 having an opportunity for strategy revision, the system will revert to state

[8] Even if a mutant is 1 or M, player 2 and $M-1$ will not change their actions because they have two neighbors who play B.

$\vec{A}$. Suppose that in $\vec{A}$, player $i \in \mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ mutates and changes his action from A to B. Player 0 with an opportunity for strategy revision will not change his action because he has $M-1$ or M neighbors who play A (we consider the case of $M \geqslant 3$). Each player in $\{\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}\} \setminus \{i\}$ with an opportunity for strategy revision will not change his action because each player in $\mathcal{I}_{c_1} \setminus \{i\}$ has two or three neighbors who play A and each player in $\mathcal{I}_{c_2} \setminus \{i\}$ has one or two neighbors who play A. With player i having an opportunity for strategy revision, the system will revert to state $\vec{A}$. Hence, $C(\vec{A}, \vec{B}) > 1$.

$\underline{C(\vec{A}, \vec{C}) > 1}$: This proof is the same as the proof of $C(\vec{A}, \vec{B}) > 1$.

$\underline{C(\vec{B}, \vec{A}) > 1}$: Suppose that in $\vec{B}$, player 0 mutates and changes his action from B to A. Each player in $\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ with an opportunity for strategy revision will not change his action because each player has two neighbors who play B. With player 0 having an opportunity for strategy revision, the system will revert to state $\vec{B}$. Suppose that in $\vec{B}$, player $i \in \mathcal{I}_{c_1} \setminus \{1, M\}$ mutates and changes his action from B to A. Player 0 with an opportunity for strategy revision will not change his action because he has $M-1$ neighbors who play B (we consider the case of $M \geqslant 3$). Each player in $\{\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}\} \setminus \{i\}$ with an opportunity for strategy revision will not change his action because each player in $\mathcal{I}_{c_1} \setminus \{i\}$ has two or three neighbors who play B and all neighbors of each player in $\mathcal{I}_{c_2} \setminus \{i\}$ play B. With player i having an opportunity for the strategy revision, the system will revert to state $\vec{B}$. Suppose that in $\vec{B}$, player $i \in \mathcal{I}_{c_2} \cup \{1, M\}$ mutates and changes his action from B to A. Player 0 and each player in $\mathcal{I}_{c_1} \setminus \{1, M\}$ will not change their actions because player 0 has $M-1$ or M neighbors who play B and each player in $\mathcal{I}_{c_1} \setminus \{1, M\}$ has two or three neighbors who play B. The system will lead the state to $\vec{B}$ or $\vec{C}$ (see the proof of $C(\vec{B}, \vec{C}) = 1$). Hence, $C(\vec{B}, \vec{A}) > 1$.

$\underline{C(\vec{C}, \vec{B}) > 1}$: It is easy to observe that even if players in $\mathcal{I} \setminus \mathcal{I}_{c_2}$ mutate, state $\vec{B}$ will not be reached. Suppose that in $\vec{C}$, player $M+2$ (player $N-1$) mutates and changes his action from A to B. Player $M+3$ (player $N-2$) with an opportunity for strategy revision will not change his action because player $M+4$ (player $N-3$) plays A (we consider the case of $M \leqslant N-4$). The system will revert to state $\vec{C}$. Suppose that each player in $\mathcal{I}_{c_2} \setminus \{M+2, N-1\}$ mutates and changes his action from A to B. Each player except the mutant will not change his action because each player in $\mathcal{I}_{c_2}$ has one or two neighbors who play A. With the mutant having an opportunity for strategy revision, the system will revert to state $\vec{C}$. Hence, $C(\vec{C}, \vec{B}) > 1$.

We can observe that the $\vec{A}$-tree of $\vec{B} \to \vec{C} \to \vec{A}$ realizes a cost of 2. By the nature of $C(\omega, \omega')$, this is the minimum cost. It is easy to observe that $\vec{B}$-tree and $\vec{C}$-tree cannot realize cost of 2. Hence, in $\tilde{G}_m(M, N+1)$, $\vec{A}$ is the unique long-run stochastically stable state.

(proof for the second part) First, we show that $C(\vec{A}, \vec{C})$, $C(\vec{B}, \vec{C})$, $C(\vec{C}, \vec{A}) = 1$ and $C(\vec{A}, \vec{B})$, $C(\vec{B}, \vec{A})$, $C(\vec{C}, \vec{B}) > 1$. The proofs of $C(\vec{B}, \vec{C}) = 1$, $C(\vec{C}, \vec{A}) = 1$, $C(\vec{B}, \vec{A}) > 1$, and $C(\vec{C}, \vec{B}) > 1$ are the same as the proof pertaining to $\tilde{G}_m(M, N+1)$.

$\underline{C(\vec{A},\vec{C}) = 1}$: Suppose that in $\vec{A}$, player $i \in \mathcal{I}_{c_1}$ mutates and changes his action from A to B. Further, suppose that player 0 has an opportunity for strategy revision, while i does not. The system will then reach the state wherein players 0 and i play B in the next period. Furthermore, suppose that i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision, while i does not. The system will then attain the state wherein players 0, i, and i's neighbors play B in the next period. Also, suppose that the neighbors of i's neighbors in $\mathcal{I}_{c_1}$ have opportunities for strategy revision. In that case, they will change their actions from A to B. By repeating this argument, it can be shown that the system will attain state $\vec{C}$. Hence, $C(\vec{A},\vec{C}) = 1$.

$\underline{C(\vec{A},\vec{B}) > 1}$: By the proof of $C(\vec{A},\vec{C}) = 1$, it is easy to observe that if player $i \in \mathcal{I}_{c_1}$ mutates, the system will attain state $\vec{A}$ or $\vec{C}$ (players in $\mathcal{I}_{c_2}$ will not change their actions because they have one or two neighbors who play A). Suppose that in $\vec{A}$, player 0 mutates and changes his action from A to B. Each player in $\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}$ will not change his action because each player has two neighbors who play A. With player 0 having an opportunity for strategy revision, the system will revert to state $\vec{A}$. Suppose that in $\vec{A}$, player $i \in \mathcal{I}_{c_2}$ mutates and changes his action from A to B. Player 0 with an opportunity for strategy revision will not change his action because all his neighbors play A. Each player in $\{\mathcal{I}_{c_1} \cup \mathcal{I}_{c_2}\} \setminus \{i\}$ with an opportunity for strategy revision will not change his action because each player in $\mathcal{I}_{c_1} \setminus \{i\}$ has two or three neighbors who play A and each player in $\mathcal{I}_{c_2} \setminus \{i\}$ has one or two neighbors who play A. With player i having an opportunity for strategy revision, the system will revert to state $\vec{A}$. Hence, $C(\vec{A},\vec{B}) > 1$.

We can observe that the $\vec{A}$-tree of $\vec{B} \to \vec{C} \to \vec{A}$ and the $\vec{C}$-tree of $\vec{A} \to \vec{C} \leftarrow \vec{B}$ achieve the cost 2. By the nature of $C(\omega,\omega')$, this is the minimum cost. It is easy to observe that $\vec{B}$-tree cannot achieve the cost 2. Hence, in $\tilde{G}_f(M, N+1)$, $\vec{A}$ and $\vec{C}$ are the long-run stochastically stable states. □

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