Knightian Uncertainty and Poverty Trap in a Model of Economic Growth*

By Shin-ichi Fukuda (University of Tokyo)**

Abstract

This paper explores how Knightian uncertainty affects dynamic properties in a model of economic growth. The decision-making theory in the analysis is that of expected utility under a non-additive probability measure, that is, the Choquet expected utility model of preference. We apply this decision theory to an overlapping-generations model where producers face uncertainty in their technologies. When the producer has aversion to uncertainty, the firm's profit function may not be differentiable. The firm's decision to invest and hire labor therefore becomes rigid for some measurable rage of real interest rate. In the dynamic equilibrium, the existence of the firm level rigidity causes discontinuity in the wage function, which makes multiple equilibria more likely outcome under log utility and Cobb-Douglass production functions. We show that even if aversion to uncertainty is small, "poverty trap" can arise for a wide range of parameter values.

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** Mailing address: Faculty of Economics, University of Tokyo, 7-3-1 Hongo Bunkyo-ku, Tokyo 113-0033, Japan. e-mail address: sfukuda@e.u-tokyo.ac.jp

1. Introduction

At the early stage of development, the degree of “uncertainty” in the economy is extremely large. Several previous empirical studies reported enormous impacts of “uncertainty” on underdeveloped economies.[1] Under a large degree of “uncertainty”, it is almost impossible to assign subjective probabilities to each unknown events. A distinction between quantifiable “risks” and unknown “uncertainty” is thus important in analyzing an engine of economic growth in underdeveloped economies.

The purpose of this paper is to show that the existence of Knightian uncertainty makes “poverty trap” more likely outcome in a standard growth model. A large number of previous theoretical studies constructed models of development traps based on multiple equilibria in physical capital accumulation (such as Barro and Becker [1989]; Murphy, Shleifer, and Vishny [1989]; Becker, Murphy, and Tamura [1990]; Zilibotti [1995]; and Galor and Weil [1996]) or in human capital accumulation (such as Azariadis and Drazen [1990]; Tsiddon [1992]; Galor and Zeira [1993]; Durlauf [1993, 1996]; Bénabou [1996]; and Galor and Tsiddon [1997]). Other models have shown various forms of indeterminacy in growth models (such as Benhabib and Farmer [1994]; Benhabib and Galí (1995); Evans, Honkapohja, and Romer [1998]) and growing through cycles (such as Matsuyama [1999, 2001]). However, except for studies such as Acemoglu and Zilibotti (1997), few studies attributed a source of the development traps to the existence of “risk” and “uncertainty” in less developed countries.

In the following analysis, we explore how Knightian uncertainty affects dynamic properties in a standard growth model. The decision-making theory we use in the analysis is that of expected utility under a nonadditive probability measure, that is, the Choquet expected utility model of preference, developed by Gilboa (1987) and Schmeidler (1989).[2] We apply the Choquet expected decision theory to an overlapping-generations model with productive capital developed by Diamond (1965). Throughout the paper, we assume log utility and Cobb-Douglas production functions. The equilibrium dynamics thus leads to unique steady state equilibrium when producers have no aversion to Knightian uncertainty. However, when the producer has aversion to Knightian uncertainty, the firm’s profit function may not be differentiable. The firm’s decision to invest and hire labor therefore becomes rigid for some measurable range of real interest rate. In the dynamic equilibrium, the rigidity is never observed. The existence of

[1] Some recent contributions on this topic are included in Collier and Pattillo eds. (2000).
[2] Based on the Gilboa-Schmeidler’s axioms, studies such as Dow and Werlang (1992), Epstein and Wang (1994), Hansen and Sargent (2000), and Mukerji and Tallon (2004) incorporate Knightian uncertainty in economic models.

the firm level rigidity, however, causes a discontinuous jump in wages, which makes multiple steady state equilibria and development trap more likely outcome. The discontinuous jump in wages reflects excessively low wages in underdeveloped economies. In underdeveloped economies, capital stock is scarce and labor supply is abundant, so that a possible decline in marginal product of labor may cause a large reduction of output. Entrepreneurs with uncertainty aversion therefore tend to reduce wage payments excessively, which in turn leads to excess decline of savings for future capital accumulation and may cause poverty traps.

The paper proceeds as follows. Section 2 sets up our basic model and explains its information structure. After formulating the expectations under Knightian uncertainty, section 3 shows the condition under which output rigidity exists for some measurable range of real interest rate. Section 4 elaborates the result by two special cases. After explaining the consumer behavior and the dynamic equilibrium in section 5, section 6 investigates the steady state and its stability. Section 7 extends the model to the case where different parameter is uncertain and investigates its stability. Section 8 summarizes our main results.

2. The Structure of Production

In the following analysis, we consider a competitive world where economic activity is performed over infinite discrete time. The structure is based on an overlapping-generations model with productive capital developed by Diamond (1965). There is a single final good, taken as a numeraire; it is competitively produced and can be either consumed or invested. Labor and capital are combined with a Cobb-Douglass technology in each firm.

Denote output per worker of firm h in period t by $y_{h,t}$ and the input of capital per worker of firm h in period t by $k_{h,t}$. The production function of firm h is then specified as

$$(1) \quad y_{h,t} = \delta k_{h,t}^{\,a_{h,t}}$$

where $0 < a_{h,t} < 1$ and $\delta > 0$.

Until section 8, we assume that δ is common for all firms and constant over time. We, however, introduce "uncertainty" in the parameter $a_{h,t}$ which is idiosyncratic across firms. At the beginning of the period, the nature assigns $a_{h,t}$ to one of n specific values A_j's (j = 1, 2, …, n) randomly, where $0 < A_1 < A_2 < \ldots < A_n < 1$. Each stochastic parameter $a_{h,t}$ is independently identically distributed over time. When deciding labor and capital inputs, each producer cannot observe the realized value of $a_{h,t}$. The producer thus faces "uncertainty" in its productivity when making production decision in each period. We assume that there is no insurance market that removes the idiosyncratic "uncertainty".

The profit per worker of firm h in period t is equal to

(2) $\Pi(k_{h,t}) = \delta k_{h,t}^{a_{h,t}} - R_t k_{h,t} - w_t$,

where R_t = real interest rate in period t and w_t = real wage in period t. Since the realized value of $a_{h,t}$ is unknown, the producer needs to maximize its "expected" profits.

What makes the following analysis distinctive from standard profit maximization problem is that we characterize the profit maximization of the producer by the Choquet expectation (see Schmeidler (1989)). Let Ω be a state space, and let $\Gamma(\Omega)$ denote the set of all subsets of Ω. Then, a convex probability capacity (or a convex non-additive probability function) is defined as a function $\theta: \Gamma(\Omega) \rightarrow [0, 1]$ which satisfies $\theta(\phi) = 0$, $\theta(\Omega) = 1$, $F \subseteq G \Rightarrow \theta(F) \leq \theta(G)$ for all F, $G \subseteq \Omega$, and $\theta(F \cup G) + \theta(F \cap G) \geq \theta(F) + \theta(G)$ for all $F, G \subseteq \Omega$.[3] Suppose that $\pi(\alpha, k)$ be a profit function such that $\pi(\cdot, k)$ is Borel-measurable for all $k \in \Re$ and that $\pi(\alpha, \cdot)$ is a differentiable concave function for all $\alpha \in \Omega$. Then, when θ is a convex probability capacity, the Choquet expected value of a random variable $\pi(\alpha, k)$ is defined by the following Choquet integral:

(3) $E_Q\, \pi(\alpha, k) \equiv \int \pi(\alpha, k)\theta(d\alpha)$,

$$= \int_0^{+\infty} \theta(\{\alpha \mid \pi(\alpha,k) \geq y\})dy + \int_{-\infty}^{0} \left[\theta(\{\alpha \mid \pi(\alpha,k) \geq y\}) - 1\right] dy,$$

whenever these integrals exist in the improper Riemann sense and are finite. In particular, if Ω is a finite state space such that $\Omega = \{\alpha_i\}_{i=1}^{n}$ and $\pi(\alpha, k)$: $S \times \Re \rightarrow \Re_+$ is a profit function such that $\pi(\alpha_1, k) \geq \pi(\alpha_2, k) \geq \cdots \geq \pi(\alpha_n, k) \geq 0$, it holds that

(4) $E_Q\, \pi(\alpha,k) = \sum_{i=1}^{n-1} \left[\pi(\alpha_i, k) - \pi(\alpha_{i+1}, k)\right] \theta(\cup_{j=1}^{i} \alpha_j) + \pi(\alpha_n, k)$.

where n is the number of outcomes of α.

[3] It is additive, and therefore a probability measure if the last inequality is always satisfied as equality.

To distinguish it from standard expectation operator, we defined the Choquet expectation operator under Knightian uncertainty by E_Q.[4] Having aversion to Knightian uncertainty, each producer chooses its capital per worker $k_{h,t}$ so as to maximize

$$(5)\quad E_Q\,\Pi(k_{h,t}) = E_Q\,[\delta\, k_{h,t}^{\ a_{h,t}} - R_t\, k_{h,t} - w_t],$$
$$= \delta\, E_Q\, k_{h,t}^{\ a_{h,t}} - R_t\, k_{h,t} - w_t.$$

The expectations are based on information available at the beginning of each period. Since both bond and labor markets are cleared before observing the realized value of $a_{h,t}$, neither real interest rate R_t nor real wage w_t is uncertain for the producer.

3. The Profit Maximization under Knightian Uncertainty

(i) The Case of Output Rigidity

To the extent that the producer has aversion to Knightian uncertainty, $E_Q\Pi(k_{h,t})$ is not differentiable in $k_{h,t}$. It holds that

$$(6)\quad \partial E_Q \Pi(k_{h,t})/\partial k_{h,t}\Big|_{k_{h,t}=1-0} = -E_Q\left[-\partial\Pi(k_{h,t})/\partial k_{h,t}\Big|_{k_{h,t}=1-0}\right]$$
$$= -\,\delta\, E_Q(-\,a_{h,t}) - R_t,$$

and

$$(7)\quad \partial E_Q \Pi(k_{h,t})/\partial k_{h,t}\Big|_{k_{h,t}=1+0} = E_Q\left[\partial\Pi(k_{h,t})/\partial k_{h,t}\Big|_{k_{h,t}=1-0}\right]$$
$$= \delta\, E_Q\, a_{h,t} - R_t.$$

(see Appendix 1 for its proof). We can then derive the following proposition.

Proposition 1: $E_Q\Pi(k_{h,t})$ is maximized at $k_{h,t} = 1$ if and only if

$$(8)\quad \delta\, E_Q\, a_{h,t} \le R_t \le -\,\delta\, E_Q\,(-a_{h,t}).$$

[4] We know that for two random variables π_1 and π_2, (i) $\pi_1 \ge \pi_2 \Rightarrow E_Q\,\pi_1 \ge E_Q\,\pi_2$, (ii) $E_Q\,(\pi_1 + \pi_2) \ge E_Q\,\pi_1 + E_Q\,\pi_2$, (iii) $-\,E_Q\,(-\pi_1) \ge E_Q\,\pi_1$, (iv) $\forall \lambda \ge 0$ and $\rho \in \Re$, $E_Q\,(\lambda\pi_1 + \rho) = \lambda E_Q\,\pi_1 + \rho$, and (v) if π_1 is strictly concave in k, $E_Q\,\pi_1$ is strictly concave in k.

Proof. Because of (6) and (7), it holds that $E_Q\left[\partial\Pi(k_{h,t})/\partial k_{h,t}\big|_{k_{h,t}=1-0}\right] \geq 0$ and $\partial E_Q\Pi(k_{h,t})/\partial k_{h,t}\big|_{k_{h,t}=1+0} \leq 0$ when (8) holds. Since $E_Q\Pi(k_{h,t})$ is strictly concave in $k_{h,t}$, this proves the proposition. [Q.E.D.]

Figure 1-(1) depicts $E_Q\Pi(k_{h,t})$ when the condition (8) holds. The function $E_Q\Pi(k_{h,t})$, which is not differentiable at $k_{h,t} = 1$, is maxmized at $k_{h,t} = 1$. Since $y_{h,t} = \delta$ when $k_{h,t} = 1$, the profit maximizing output depends neither on parameter $a_{h,t}$ nor on real interest rate R_t when $k_{h,t} = 1$. The above proposition therefore states that if (8) holds, the model can have "output rigidity" under Knightian uncertainty. In the absence of Knightian uncertainty, $E_Q\, a_{h,t} = -\, E_Q\,(-a_{h,t})$, so that there exists no measurable range of $k_{h,t}$ that satisfies (8). However, when the firm has aversion to Knightian uncertainty, $-\, E_Q\,(-a_{h,t}) > E_Q\, a_{h,t}$. Our model can thus have some measurable range of R_t that satisfies (8) under some circumstances.

(ii) The Case when Output is not Rigid

When (8) does not hold, the profit maximizing output is no longer rigid in the sense that it is determined by relevant parameters and by real interest rate. The determinants of $k_{h,t}$, however, differ depending on whether $k_{h,t}$ is greater than 1 or less than 1. From (4), it holds that

(9a) $E_Q\Pi(k_{h,t}) = \delta\,[\,\Sigma_{i=1}^{n-1}(k_{h,t}^{\ A_i} - k_{h,t}^{\ A_{i+1}})\,\theta(\,\mathrm{U}_{j=1}^{i}A_j) + k_{h,t}^{\ A_n}\,] - R_t\, k_{h,t} - w_t$, when $k_{h,t} < 1$,

and

(9b) $E_Q\Pi(k_{h,t}) = \delta\,[\,\Sigma_{i=1}^{n-1}(k_{h,t}^{\ A_{n+1-i}} - k_{h,t}^{\ A_{n-i}})\,\theta(\,\mathrm{U}_{j=1}^{i}A_{n+1-j}) + k_{h,t}^{\ A_1}\,] - R_t\, k_{h,t} - w_t$, when $k_{h,t} > 1$,

where θ is a convex probability capacity. We can thus derive the following proposition.

Proposition 2: When $R_t > -\delta\, E_Q(-a_{h,t})$, the profit maximizing capital stock $k_{h,t}$ is less than 1 and satisfies $\partial E_Q\Pi(k_{h,t})/\partial k_{h,t} = 0$ in (9a). On the other hand, when $R_t < \delta\, E_Q\, a_{h,t}$, the profit maximizing capital stock $k_{h,t}$ is greater than 1 and satisfies $\partial E_Q\Pi(k_{h,t})/\partial k_{h,t} = 0$ in (9b).

Proof. Because of (9a) and (9b), it holds that $E_Q\left[\partial\Pi(k_{h,t})/\partial k_{h,t}\big|_{k_{h,t}=1-0}\right] < 0$ when $R_t > -\delta$

$E_Q(-a_{h,t})$ and that $\partial E_Q \Pi(k_{h,t}) / \partial k_{h,t} \big|_{k_{h,t}=1+0} > 0$ when $R_t < \delta E_Q(a_{h,t})$. Since both (9a) and (9b) are differentiable in $k_{h,t}$, we obtain the proposition. [Q.E.D.]

Figure 1-(2) depicts $E_Q\Pi(k_{h,t})$ when $R_t > -\delta E_Q(-a_{h,t})$. The function $E_Q\Pi(k_{h,t})$ is maximized at $k_{h,t} = k^* < 1$. It is noteworthy that the equilibrium capital stock per worker $k_{h,t}$ is identical for all firms. This is because each firm is identical ex ante in the sense that it decides its capital stock and labor input before observing idiosyncratic shock $a_{h,t}$.

4. Two Special Cases

(i) The Case of Two States of Nature

The decision theory can be formulated in a tractable form for the case of two states of nature, that is, when $n = 2$. This corresponds to the case where parameter $a_{h,t}$ takes either A_1 or A_2, where $0 < A_1 < A_2 < 1$. Suppose that $a_{h,t} = A_1$ in state 1 and $a_{h,t} = A_2$ in state 2 where $0 < A_1 < A_2 < 1$. Let $S = \{A_1, A_2\}$, and assume that $\theta(\{A_1\}) = \theta(\{A_2\}) = \nu \leq 0.5$, $\theta(\phi) = 0$, and $\theta(S) = 1$, where θ is a convex probability capacity. It is easy to see that θ is a probability measure when $\nu = 0.5$. One can thus interpret that there exists Knightian uncertainty if and only if $0 \leq \nu < 0.5$. In particular, if we follow the definition of "uncertainty aversion" by Dow and Werlang (1992), we can see that uncertainty aversion is larger when ν is smaller.

Since $1-A_1 > 1-A_2$, (4) leads to

(10a) $E_Q(1-a_{h,t}) = \nu(1-A_1) + (1-\nu)(1-A_2)$,

(10b) $E_Q(-1+a_{h,t}) = (1-\nu)(1-A_1) + \nu(1-A_2)$.

The condition (8) is thus written as

(11) $\delta[(1-\nu)A_1 + \nu A_2] \leq R_t \leq \delta[\nu A_1 + (1-\nu)(1-A_2)]$.

From proposition 1, the profit maximizing capital stock per worker $k_{h,t}$ is independent of real shocks when the condition (11) holds. The condition (11) is never satisfied when $\nu = 0.5$, that is, when there exists no Knightian uncertainty. However, to the extent that $\nu < 0.5$, some algebraic arrangements verify that the condition (11) is satisfied for a measurable range of real interest rate, particularly when A_2 is large or when ν is small.

In the case of two states of nature, $E_Q\Pi(k_{h,t})$ is equal to

(12a) $E_Q\Pi(k_{h,t}) = \delta[\nu\, k_{h,t}^{\;A_1} + (1\text{-}\nu)\, k_{h,t}^{\;A_2}] - R_t\, k_{h,t} - w_t$ when $k_{h,t} < 1$,

and

(12b) $E_Q\Pi(k_{h,t}) = \delta[(1\text{-}\nu)\, k_{h,t}^{\;A_1} + \nu\, k_{h,t}^{\;A_2}] - R_t\, k_{h,t} - w_t$ when $k_{h,t} > 1$,

When $k_{h,t} \neq 1$, the profit maximizing capital stock per worker is thus obtained by differentiating one of these equations.

(ii) The Case of Maxi-min Rule under Complete Ignorance

The maxi-min rule under complete ignorance proposed by Wald (1950) is a special case of the decision theory under Knightian uncertainty. Under the Wald's maxi-min rule, a person with extreme uncertainty aversion who is completely uninformed maximizes the payoff of the worst possible outcome.

Since $0 < A_1 < A_2 < \ldots < A_n < 1$, $E_Q\, a_{h,t} = A_1$ and $E_Q\,(-a_{h,t}) = -1/A_n$ under the Wald's maxi-min rule. The condition (8) is then written as

(13) $\delta A_1 \leq R_t \leq \delta A_n$.

Under the Wald's maxi-min rule, $E_Q\Pi(k_{h,t})$ becomes

(14a) $\Pi^A(k_{h,t}) \equiv \delta\, k_{h,t}^{\;A_n} - R_t\, k_{h,t} - w_t$ when $k_{h,t} \leq 1$,

(14b) $\Pi^B(k_{h,t}) \equiv \delta\, k_{h,t}^{\;A_1} - R_t\, k_{h,t} - w_t$ when $k_{h,t} > 1$.

We thus obtain the equilibrium capital stock per worker as

(15a) $k_{h,t} = 1$ when (13) holds,

(15b) $k_{h,t} = (\delta A_n/R_t)^{1/(1-A_n)}$ when $R_t > \delta A_n$,

(15c) $k_{h,t} = (\delta A_1/R_t)^{1/(1-A_1)}$ when $R_t < \delta A_1$.

Since $0 < A_1 < A_n < 1$, $k_{h,t}$ is less than one when $R_t > \delta A_n$ and is greater than one when $R_t < \delta A_1$.

5. The Saving Behavior and the Equilibrium Dynamics

In our overlapping generations model, identical individuals are born in every period t. For simplicity, population of individuals is assumed to be constant across generations. Individuals live for two periods. In the first they work and earn the competitive market wage w_t, and in the second they retired. Let c_{1t} and c_{2t} denote the consumption in period t of young and old individuals. Individuals born in period t are characterized by the following intertemporal utility function

$$(16)\quad U_t = \ln c_{1t} + \beta \ln c_{2t+1},$$

where $0 < \beta$ is the discount factor.

Each individual supplies one unit of labor when he or she is young and divides the resulting labor income w_t between first-period consumption c_{1t} and saving s_t. The saving consists of two types. One is bond that earns risk-free return R_{t+1} in the following period. The other is stock that earns dividend and capital gain (or loss). The returns from these two types of savings enable the cohort to consume during the retirement. Suppose that ω is the share of risk-free bonds in the saving. The intertemporal budget constraint is then written as

$$(17a)\quad c_{1t} + s_t = w_t,$$
$$(17b)\quad c_{2t} = [(1+ R_{t+1})\omega + (1+ r_{t+1})(1-\omega)]\, s_t,$$

where r_{t+1} is returns for stock holdings from period t to t+1.

Since both bond and labor markets are cleared before observing the realized value of $a_{h,t}$, the consumer faces any uncertainty neither in real interest rate R_{t+1} nor in real wage w_t. Because of the law of large number, the consumer can also diversity idiosyncratic uncertainty in r_{t+1}. The arbitrage condition thus implies that $R_t = r_t$ for all t. The consumer maximizes the intertemporal utility function (16) subject to the budget constraint (17a). Assuming the interior solutions, the first-order conditions are

$$(18)\quad s_t = [\beta/(1+\beta)]w_t.$$

Equation (18) denotes the saving function. Because of log utility, the saving function is independent of the saving components and their rates of returns.

Denote capital stock per worker in the economy in period t by k_t. Then, since the capital stock in period t+1 is the amount saved by young individuals in period t, it holds that $k_{t+1} = s_t$. We can thus obtain

$$(19)\quad k_{t+1} = [\beta/(1+\beta)]\, w_t.$$

Equation (19) characterizes the equilibrium dynamics in our macroeconomic model. Since capital and labor are competitively priced before observing the realized value of $a_{h,t}$, the representative firm's Choquet expected profit maximization determines the equilibrium value of R_t and w_t. Proposition 2 in section 3 suggests that when $0 < k_t < 1$ or $k_t > 1$,

$$(20)\quad R_t = \delta\, \partial E_Q\, k_t^{\,a_{h,t}} / \partial k_t,$$

$$(21)\quad w_t \equiv w(k_t) = \delta\, E_Q\, k_t^{\,a_{h,t}} - k_t\, \delta\, \partial E_Q\, k_t^{\,a_{h,t}} / \partial k_t.$$

Proposition 1, in contrast, implies that when $k_t = 1$, R_t satisfies the condition (8) and w_t is determined by $w_t = E_Q\, k_t^{\,a_{h,t}} - R_t\, k_t$.

Since the definition (4) implies that

$$(22a)\quad E_Q\, k_t^{\,a_{h,t}} = \Sigma_{i=1}^{n-1} (k_t^{A_i} - k_t^{A_{i+1}})\,\theta(\mathrm{U}_{j=1}^{i} A_j) + k_t^{A_n} \quad \text{when } 0 < k_t < 1,$$

$$(22b)\quad E_Q\, k_t^{\,a_{h,t}} = 1 \text{ when } k_t = 1,$$

$$(22c)\quad E_Q\, k_t^{\,a_{h,t}} = \Sigma_{i=1}^{n-1} (k_t^{A_{n+1-i}} - k_t^{A_{n-i}})\,\theta(\mathrm{U}_{j=1}^{i} A_{n+1-j}) + k_t^{A_1} \quad \text{when } k_t > 1,$$

where θ is a convex probability capacity, we therefore obtain

$$(23a)\quad w_t = \delta(1-A_1)\, k_t^{A_1}\, \theta(A_1) + \delta\Sigma_{i=2}^{n-1} (1-A_i)\, k_t^{A_i}\, [\theta(\mathrm{U}_{j=1}^{i} A_j) - \theta(\mathrm{U}_{j=1}^{i-1} A_j)]$$

$$+\ \delta(1-A_n)\, k_t^{A_n}\, [1-\theta(\mathrm{U}_{j=1}^{n-1} A_j)] \qquad \text{when } 0 \le k_t < 1,$$

$$(23b)\quad \delta[1+E_Q(-a_{h,t})] \le w_t \le \delta(1-E_Q\, a_{h,t}) \qquad \text{when } k_t = 1,$$

$$(23c)\quad w_t = \delta(1-A_1)\, k_t^{A_1}\, [1-\theta(\mathrm{U}_{j=1}^{n-1} A_{n+1-j})]$$

$$+\ \delta\Sigma_{i=2}^{n-1} (1-A_{n+1-i}) k_t^{A_{n+1-i}}\, [\theta(\mathrm{U}_{j=1}^{i} A_{n+1-j}) - \theta(\mathrm{U}_{j=1}^{i-1} A_{n+1-j})]$$

$$+\ \delta(1-A_n) k_t^{A_n}\, \theta(A_n) \qquad \text{when } k_t > 1,$$

where $E_Q(-a_{h,t}) \equiv -\ \Sigma_{i=1}^{n-1} A_i\, [\theta(\mathrm{U}_{j=1}^{i} A_j) - \theta(\mathrm{U}_{j=1}^{i-1} A_j)] - A_n$ and $E_Q\, a_{h,t} \equiv \Sigma_{i=1}^{n-1} A_{n+1-i}\, [\theta(\mathrm{U}_{j=1}^{i} A_{n+1-j}) -$

$\theta(\cup_{j=1}^{i-1} A_{n+1-j})] + A_n$.

It is noteworthy that inequalities in (23b) provide a mirror image of Proposition 1. They imply that k_t remains to be one not only for a measurable range of R_t but also for a measurable range of w_t in equilibrium. From (23a,b,c), we can derive the following proposition.

Proposition 3: The function $w(k_t)$ satisfies the following four properties.

(I) $w(k_t)$ is strictly increasing and strictly concave in k_t when $0 \leq k_t < 1$.

(II) $w(k_t)$ is strictly increasing and strictly concave in k_t when $k_t > 1$.

(III) $w(0) = 0$, $\lim_{k\to 0} w'(k_t) = +\infty$, and $\lim_{k\to +\infty} w'(k_t) = 0$.

(IV) $\lim_{k\to 1-0} w(k_t) = \delta[1+E_Q(-a_{h,t})] < \delta(1-E_Q a_{h,t}) = \lim_{k\to 1+0} w(k_t)$.

Proof: Since $\theta(A_1) > 0$, $\theta(A_n) > 0$, $\theta(\cup_{j=1}^{i} A_j) \geq \theta(\cup_{j=1}^{i-1} A_j)$, $\theta(\cup_{j=1}^{i} A_{n+1-j}) > \theta(\cup_{j=1}^{i-1} A_{n+1-j})$, $1 > \theta(\cup_{j=1}^{n-1} A_j)$, and $1 > \theta(\cup_{j=1}^{n-1} A_{n+1-j})$, we can derive (I), (II), and (III) from equations (23a) and (23c). Moreover, equations (23a) and (23c) respectively lead to

$$\lim_{k\to 1-0} w(k_t) = \delta(1-A_1)\theta(A_1) + \delta\Sigma_{i=2}^{n-1}(1-A_i)[\theta(\cup_{j=1}^{i} A_j) - \theta(\cup_{j=1}^{i-1} A_j)]$$

$$+ \delta(1-A_n)[1-\theta(\cup_{j=1}^{n-1} A_j)] = \delta[1+E_Q(-a_{h,t})],$$

$$\lim_{k\to 1+0} w(k_t) = \delta(1-A_1)[1-\theta(\cup_{j=1}^{n-1} A_{n+1-j})] + \delta\Sigma_{i=2}^{n-1}(1-A_{n+1-i})[\theta(\cup_{j=1}^{i} A_{n+1-j})-\theta(\cup_{j=1}^{i-1} A_{n+1-j})]$$

$$+\delta(1-A_n)\theta(A_n) = \delta(1-E_Q a_{h,t}).$$

This proves (IV) because $-E_Q(-a_{h,t}) > E_Q a_{h,t}$. [Q.E.D.]

Among four properties in Proposition 3, (I), (II), and (III) hold true even when there exists no Knightian uncertainty. However, unless there exists Knightian uncertainty, (IV) never holds because $-E_Q(-a_{h,t}) = E_Q a_{h,t}$ when θ is a probability measure. Figure 2 depicts the wage function $w(k_t)$ that satisfies four properties in Proposition 3. Because of (IV), $w(k_t)$ is not continuous at $k_t = 1$. Around $k_t = 1$, wages can be either low or high, depending on which side of the critical level one is coming from. The lower wages when $k_t < 1$ are the source of poverty traps in the following analysis.

6. The Steady Sate and the Stability

Because of (I), (II), and (IV) in Proposition 3, the function $w(k_t)$ is well defined over all $k_t \geq 0$. The equilibrium dynamics of k_t is thus described by the following one-dimensional map:

$$(24) \quad k_{t+1} = \lambda\, w(k_t).$$

where $\lambda \equiv \{\beta/[(1+n)(1+\beta)]\}$.

Since $w(k_t)$ is piecewise continuous for all $k_t \geq 0$, (III) in Proposition 3 implies that the one-dimensional map always has at least one positive steady state equilibrium k^*. However, because of (IV) in Proposition 3, the non-zero steady state equilibrium k^* is not necessarily unique under Knightian uncertainty. The following proposition holds.

Proposition 4:

(I) If $\lim_{k\to 1-0} w(k_t) > 1/\lambda$, there exists unique and globally stable k^* such that $k^* > 1$.

(II) If $\lim_{k\to 1+0} w(k_t) < 1/\lambda$, there exists unique and globally stable k^* such that $k^* < 1$.

(III) If $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$, there exist three non-zero steady state equilibria k_1^*, 1, and k_2^* such that $0 < k_1^* < 1 < k_2^*$. k_t converges to k_1^* when $0 < k_t < 1$ and k_t converges to k_2^* when $k_t > 1$. The steady state equilibrium $k^* = 1$ is locally unstable.

Proof: Using four properties in Proposition 3, we can show (I), (II), and (III) in the above proposition graphically. First, Figure 3-(I) depicts the one-dimensional map for the case when $\lim_{k\to 1-0} w(k_t) > 1/\lambda$. In the figure, the map never intersects the 45 degree line when $0 \leq k_t \leq 1$, while it has one intersection with the 45 degree line when $k_t > 1$. Since $k_{t+1} > k_t$ when $k_t < k^*$ and $k_{t+1} < k_t$ when $k_t > k^*$, the non-zero steady state equilibrium k^* is globally stable in Figure 3-(I). This verifies (I) in Proposition 4.

Figure 3-(II) depicts the one-dimensional map for the case when $\lim_{k\to 1+0} w(k_t) < 1/\lambda$. Equation (23c) implies that $\lim_{k\to 1+0} \lambda\, w'(k_t) < 1$ if $\lim_{k\to 1+0} \lambda\, w(k_t) < 1$, so that $\lambda\, w(k_t) < k_t$ for all $k_t > 1$. Figure 3-(II) is thus the only possible map when $\lim_{k\to 1+0} w(k_t) < 1/\lambda$. Noting that $k_{t+1} > k_t$ when $k_t < k^*$ and $k_{t+1} < k_t$ when $k_t > k^*$, the steady state k^* is globally stable in Figure 3-(II). This verifies (II) in Proposition 4.

Finally, Figure 3-(III) depicts the one-dimensional map for the case when $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$. The map intersects the 45 degree line at $k_t = k_1{}^*$ when $0 < k_t < 1$ and at $k_t = k_2{}^*$ when $k_t > 1$. The map shows the existence of three non-zero steady state equilibria, $k_1{}^*$, 1, and $k_2{}^*$ such that $0 < k_1{}^* < 1 < k_2{}^*$. It also implies that both of the steady states $k_1{}^*$ and $k_2{}^*$ are locally stable but that the steady state $k^* = 1$ is locally unstable. This verifies (III) in Proposition 4. [Q.E.D.]

Proposition 4 has two noteworthy implications. One is that output rigidity where $k_t = 1$ is never observed in our dynamic equilibrium. If $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$, output rigidity arises only at a locally unstable steady state equilibrium. In other cases, output rigidity can arise during the transition path but it is realized only for countable initial values. The existence of unobservable output rigidity, however, causes a jump of real wage w_t around $k^* = 1$. For example, if $\lim_{k\to 1-0} w(k_t) > 1/\lambda$, w_t gradually increases to $\delta[1+E_Q(-a_{h,t})]$ when $k_t < 1$. But when k_t goes up beyond one, w_t has a jump above $\delta[1-E_Q\, a_{h,t}]$, followed by another gradual increases towards the steady state $k^* > 1$.

The other implication is that if $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$, there exist two locally stable steady states. The steady state equilibrium with high output arises only when the initial value of k_t is greater than one. In contrast, when the initial value of k_t is less than one, "poverty trap" arises and the economy converges to the steady state equilibrium with low output in the long-run. The general condition for which $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$ is complicated. However, in the case of two-state nature discussed in section 4(i), $\lim_{k\to 1-0} w(k_t) = \delta[(1-A_1)v + (1-A_2)(1-v)]$ and $\lim_{k\to 1+0} w(k_t) = \delta[(1-A_1)(1-v) + (1-A_2)v]$. It thus holds that $\lim_{k\to 1-0} w(k_t) < 1/\lambda < \lim_{k\to 1+0} w(k_t)$ if and only if

$$(25)\quad (1-A_1)v + (1-A_2)(1-v) < 1/(\lambda\delta) < (1-A_1)(1-v) + (1-A_2)v.$$

When there exists no Knightian uncertainty (that is, when $v = 0.5$), the condition (25) never holds. However, when v is small, the condition tends to hold for a wide set of parameter values. It is also noteworthy that even when v is close to 0.5, the condition can hold for some significantly large ranges of parameter values. For example, assuming that $A_1 = 0.65$, table 1 reports the upper and lower values of $\lambda\delta$ that satisfy the condition (25) for alternative values of A_2 and v. The table reports that the difference between the upper and lower values can be significant for various combinations of A_2 and v. In particular, it shows that when the difference between A_1 and A_2 is large, the value of $\lambda\delta$ can have sufficiently large variations even if v is close to 0.5. The results imply that the condition (25) holds for a wide range of parameter values even if aversion to Knightian uncertainty is very small.

Some economists may argue that the degree of Knightian uncertainty is, if any, very small and that the foundations of rational expectations still hold in macroeconomics approximately. The above result suggests that this criticism is irrelevant in our model. Because of the envelope theorem, the loss from output rigidity is a second order. Thus, even when v is close to 0.5, "poverty trap" can arise for some significant range of parameter values.

7. The Case where the parameter δ is uncertain

Until the last section, we have considered the case where there exists "uncertainty" only in the parameter $a_{h,t}$. The purpose of this section is to consider the dynamic properties in our overlapping generations model when the firm faces "uncertainty" in parameter δ. Even when there exists "uncertainty" in δ, equation (24) still describes the equilibrium dynamics. The properties of real wage w_t, however, become different when δ is uncertain. Denoting δ of firm h by $\delta_{h,t}$, it holds that

$$(26)\quad w_t \equiv w(k_t) = E_Q(\delta_{h,t}\, k_t^{a_{h,t}}) - k_t\, \partial E_Q(\delta_{h,t}\, k_t^{a_{h,t}})/\partial k_t.$$

When there exists uncertainty only in $\delta_{h,t}$, the dynamic property is essentially the same as that in the standard overlapping generations model in the sense that it is globally stable. However, when there exists uncertainty both in $a_{h,t}$ and $\delta_{h,t}$, the equilibrium dynamics becomes highly complicated in general. For simplicity, the following analysis focuses on the case where each of $a_{h,t}$ and $\delta_{h,t}$ takes one of two distinct values A_i and D_j (j = 1, 2) respectively, where $0 < A_1 < A_2 < 1$ and $0 < D_1 < D_2$. We assume that idiosyncratic shocks in $a_{h,t}$ and $\delta_{h,t}$ are independently identically distributed over time.

Define $\kappa_0 \equiv (D_1/D_2)^{1/(A_2-A_1)}$ and $\kappa_1 \equiv (D_2/D_1)^{1/(A_2-A_1)}$. Then, we can show that real wage $w_t \equiv w(k_t)$ is determined by the following equations (see Appendix 2 for their derivations).

$$\begin{aligned}(27a)\ w(k_t) = {} & D_1 (1-A_2)\, k_t^{A_2} (1-\theta^1_{22}) + D_2 (1-A_2)\, k_t^{A_2} (\theta^1_{22}-\theta(\{A_1\})) \\ & + D_1 (1-A_1)\, k_t^{A_1} (\theta(\{A_1\})-\theta_{12}) + D_2 (1-A_1)\, k_t^{A_1}\, \theta_{12} \quad \text{when } 0 < k_t < \kappa_0,\end{aligned}$$

$$\begin{aligned}(27b)\ w(k_t) = {} & D_1 (1-A_2)\, k_t^{A_2} (1-\theta^1_{22}) + D_1 (1-A_1)\, k_t^{A_1} (\theta^1_{22}-\theta(\{\delta_2\})) \\ & + D_2 (1-A_2)\, k_t^{A_2} (\theta(\{\delta_2\})-\theta_{12}) + D_2 (1-A_1)\, k_t^{A_1}\, \theta_{12} \quad \text{when } \kappa_0 < k_t < 1,\end{aligned}$$

(27c) $w(k_t) = D_1\,(1\text{-}A_1)\,k_t^{A_1}\,(1\text{-}\theta^2_{12}) + D_1\,(1\text{-}A_2)\,k_t^{A_2}\,(\theta^2_{12}\text{-}\theta(\{\delta_2\}))$

$+ D_2\,(1\text{-}A_1)\,k_t^{A_1}\,(\theta(\delta_2)\text{-}\theta_{22}) + D_2\,(1\text{-}A_2)\,k_t^{A_2}\,\theta_{22}$ when $1 < k_t < \kappa_1$,

(27d) $w(k_t) = D_1\,(1\text{-}A_1)\,k_t^{A_1}\,(1\text{-}\theta^2_{12}) + D_2\,(1\text{-}A_1)\,k_t^{A_1}\,(\theta^2_{12}\text{-}\theta(\{A_2\}))$

$+ D_1\,(1\text{-}A_2)\,k_t^{A_2}\,(\theta(\{A_2\})\text{-}\theta_{22}) + D_2\,(1\text{-}A_2)\,k_t^{A_2}\,\theta_{22}$ when $0 < k_t < \kappa_0$.

where $\theta_{12} \equiv \theta(\{A_1, \delta_2\})$, $\theta_{22} \equiv \theta(\{A_2, \delta_2\})$, $\theta^1_{22} \equiv \theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}\cup\{A_2, \delta_2\})$, and $\theta^2_{12} \equiv \theta(\{A_1, \delta_2\}\cup\{A_2, \delta_1\}\cup\{A_2, \delta_2\})$.

We can easily see that the function $w(k_t)$ in each of the above equations is strictly increasing and strictly concave in k_t. We can also show that $w(0) = 0$, $\lim_{k\to 0} w'(k_t) = +\infty$, and $\lim_{k\to+\infty} w'(k_t) = 0$. However, when there exists uncertainty both in $a_{h,t}$ and $\delta_{h,t}$, $w(k_t)$ is not necessarily increasing in k_t. It holds that

(28a) $\lim_{k\to\kappa_0+0} w(k_t) - \lim_{k\to\kappa_0-0} w(k_t) = \psi_0\,(A_2\text{-}A_1)(\theta^1_{22}\text{-}\theta(\{A_1\})\text{-}\theta(\{\delta_2\})+\theta_{12})$,

(28b) $\lim_{k\to 1+0} w(k_t) - \lim_{k\to 1-0} w(k_t)$

$= (A_2\text{-}A_1)[\delta_1\,(1\text{-}\theta^1_{22}\text{-}\theta^2_{12} + \theta(\{\delta_2\})) + \delta_2\,(\theta(\{\delta_2\})\text{-}\theta_{12}\text{-}\theta_{22})]$,

(28c) $\lim_{k\to\kappa_1+0} w(k_t) - \lim_{k\to\kappa_1-0} w(k_t) = \psi_1\,(A_2\text{-}A_1)(\theta^2_{12}\text{-}\theta(\{A_2\})\text{-}\theta(\{\delta_2\})+\theta_{22})$,

where $\psi_0 \equiv \delta_1\,k_t^{A_1} = \delta_2\,k_t^{A_2}$ and $\psi_1 \equiv \delta_2\,k_t^{A_1} = \delta_1\,k_t^{A_2}$. We can thus obtain the following proposition.

Proposition 5:

(I) $\lim_{k\to\kappa_0-0} w(k_t) < \lim_{k\to\kappa_0+0} w(k_t)$ if and only if

(29a) $\theta^1_{22}\text{-}\theta(\{A_1\})\text{-}\theta(\{\delta_2\})+\theta_{12} > 0$,

(II) $\lim_{k\to 1-0} w(k_t) < \lim_{k\to 1+0} w(k_t)$,

(29b) $\delta_1\,(1\text{-}\theta^1_{22}\text{-}\theta^2_{12} + \theta(\{\delta_2\})) + \delta_2\,(\theta(\{\delta_2\})\text{-}\theta_{12}\text{-}\theta_{22}) > 0$,

(III) $\lim_{k\to\kappa_1-0} w(k_t) < \lim_{k\to\kappa_1+0} w(k_t)$ if and only if

(29c) $\theta^2_{12}\text{-}\theta(\{A_2\})\text{-}\theta(\{\delta_2\})+\theta_{22} > 0$.

Proof: Obvious because $A_2 > A_1$. [Q.E.D.]

All of the conditions (29a), (29b), and (29c) hold for a class of convex probability capacities. To see this, define probability measures for events $\{A_1, \delta_1\}$, $\{A_1, \delta_2\}$, $\{A_2, \delta_1\}$, and $\{A_2, \delta_2\}$ by p_{11}, p_{12}, p_{21}, and p_{22} respectively and assume that the events are independent, that is, $p_{11} + p_{12} + p_{21} + p_{22} = 1$. Then, θ is a convex probability capacity if

$$\theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}\cup\{A_2, \delta_1\}\cup\{A_2, \delta_2\}) \equiv (p_{11} + p_{12} + p_{21} + p_{22})^{1+\varepsilon} = 1, \quad \theta(\phi) = 0,$$
$$\theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}\cup\{A_2, \delta_1\}) \equiv (p_{11} + p_{12} + p_{21})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}\cup\{A_2, \delta_2\}) \equiv (p_{11} + p_{12} + p_{22})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_1\}\cup\{A_2, \delta_1\}\cup\{A_2, \delta_2\}) \equiv (p_{11} + p_{21} + p_{22})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_2\}\cup\{A_2, \delta_1\}\cup\{A_2, \delta_2\}) \equiv (p_{12} + p_{21} + p_{22})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}) \equiv (p_{11} + p_{12})^{1+\varepsilon}, \quad \theta(\{A_1, \delta_1\}\cup\{A_2, \delta_1\}) \equiv (p_{11} + p_{21})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_1\}\cup\{A_2, \delta_2\}) \equiv (p_{11} + p_{22})^{1+\varepsilon}, \quad \theta(\{A_1, \delta_2\}\cup\{A_2, \delta_1\}) \equiv (p_{12} + p_{21})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_2\}\cup\{A_2, \delta_2\}) \equiv (p_{11} + p_{22})^{1+\varepsilon}, \quad \theta(\{A_2, \delta_1\}\cup\{A_2, \delta_2\}) \equiv (p_{21} + p_{22})^{1+\varepsilon},$$
$$\theta(\{A_1, \delta_1\}) \equiv p_{11}^{1+\varepsilon}, \; \theta(\{A_1, \delta_2\}) \equiv p_{12}^{1+\varepsilon}, \; \theta(\{A_2, \delta_1\}) \equiv p_{21}^{1+\varepsilon}, \; \theta(\{A_2, \delta_2\}) \equiv p_{22}^{1+\varepsilon},$$

when $\varepsilon > 0$. We can show that all of the conditions (29a), (29b), and (29c) always hold for this convex probability capacity.

Since $w(k_t)$ is piecewise continuous for all $k_t \geq 0$, the one-dimensional map always has a positive steady state equilibrium k^* even when both $a_{h,t}$ and $\delta_{h,t}$ are uncertain. The number of non-zero steady state equilibria and their stabilities, however, depend on parameters. When all of the conditions (30a), (30b), and (30c) hold, the model has at least one locally stable non-zero steady state equilibrium. For any parameters, k_t converges to one of the stable non-zero steady state equilibria monotonically. However, depending on parameters, the model can have four types of non-zero steady state equilibria: (A) one non-zero locally steady state equilibrium k^*, (B) two locally stable steady state equilibria k_1^* and k_2^* where $k_1^* < k_2^*$, and one locally unstable non-zero steady state equilibrium, (C) three locally stable steady state equilibria k_1^* , k_2^*, and k_3^* where $k_1^* < k_2^* < k_3^*$, and two locally unstable non-zero steady state equilibria, and (D) four locally stable steady state equilibria k_1^* , k_2^*, k_3^*, and k_4^* where $k_1^* < k_2^* < k_3^* < k_4^*$, and three locally unstable non-zero steady state equilibria.

Each type can arise, depending on where the 45 degree line intersects with the curve of $\lambda\, w(k_t)$. Figure 4 depicts an example of the type (B). It is easy to see that both k_1^* and k_2^* are locally stable because $\lambda\ [\partial w(k_t)/\ \partial\ k_t] < 1$ around k_1* and k_2*. The steady state with high output k_2* arises only when the initial value of k_t is greater than some threshold value. When the initial value of k_t is less than the threshold value, "poverty trap" arises and the economy converges to the steady state with low output k_1*.

8. Concluding Remarks

In this paper, we explored how Knightian uncertainty affects dynamic properties of a standard growth model. We applied the Choquet expected decision theory to an overlapping-generations model where firms face uncertainty in technologies. To the extent that the firm has aversion to Knightian uncertainty, the firm's decision becomes rigidity for a significant range of real interest rate. The existence of the firm level rigidity causes a significant discontinuity of real wage w_t around $k^* = 1$. As a result, there exist multiple locally stable steady states under some parameters even if aversion to Knightian uncertainty is very small. When the initial value of k_t is small, "poverty trap" arises and the economy converges to the steady state equilibrium with low output in the long-run. Solving "uncertainty" is a critical problem at the early stage of development.

Appendix 1: Derivations of (6) and (7)

This appendix derives (6) and (7) in section 2 by using the following lemma.

Lemma: Suppose that for all $\alpha \in \Omega$, a profit function $\pi(\alpha, k)$, which is concave in k, is independent of α when $k = k_0$. Then, it holds that

$$\text{(A1)} \quad \partial E_Q \pi(\alpha,k)/\partial k\Big|_{k=k_0-0} = -E_Q(-\partial\pi(\alpha,k)/\partial k\big|_{k=k_0}),$$

$$\text{(A2)} \quad \partial E_Q \pi(\alpha,k)/\partial k\Big|_{k=k_0+0} = E_Q(\partial\pi(\alpha,k)/\partial k\big|_{k=k_0}).$$

Proof. Suppose that $\pi(\alpha_1, X) = \pi(\alpha_2, X) = \cdots = \pi(\alpha_n, X)$ when $k = k_0$, where n is the number of outcomes of α. Since $\pi(\alpha, k)$ is a concave function in k, we can then suppose that $\pi(\alpha_1, k) \geq \pi(\alpha_2, k) \geq \cdots \geq \pi(\alpha_n, k) \geq 0$ when $k = k_0+0$ and that $\pi(\alpha_1, k) \leq \pi(\alpha_2, k) \leq \cdots \leq \pi(\alpha_n, k) \leq 0$ when $k = k_0-0$ without loss of generality. In this case, it holds that $\partial\pi(\alpha_1, k)/\partial k \geq \partial\pi(\alpha_2, k)/\partial k \geq \cdots \geq \partial\pi(\alpha_n, k)/\partial k \geq 0$ when $k = k_0+0$. Therefore, the definition of the Choquet expectation, that is (4) in section 2, implies that

$$\partial E_Q \pi(\alpha,k)/\partial k\Big|_{k=k_0+0} = E_Q(\partial\pi(\alpha,k)/\partial k\big|_{k=k_0}).$$

Similarly, it holds that $0 \leq \partial\pi(\alpha_1, k)/\partial k \leq \partial\pi(\alpha_2, k)/\partial k \leq \cdots \leq \partial\pi(\alpha_n, k)/\partial k$ when $k = k_0-0$. The definition (14) thus implies that

$$\partial E_Q \pi(\alpha,k)/\partial k\Big|_{k=k_0-0} = -E_Q(-\partial\pi(\alpha,k)/\partial k\big|_{k=k_0}).$$

This proves the lemma. [Q.E.D.]

Applying this lemma to equation (2), we obtain (6) and (7) in section 2.

Appendix 2: Derivations of real wage in section 7

Since $\kappa_0 \equiv (D_1/D_2)^{1/(A_2-A_1)}$ and $\kappa_1 \equiv (D_2/D_1)^{1/(A_2-A_1)}$, it holds that

$D_1 k_t^{A_2} < D_2 k_t^{A_2} < D_1 k_t^{A_1} < D_2 k_t^{A_1}$ when $0 < k_t < \kappa_0$,

$D_1 k_t^{A_2} < D_1 k_t^{A_1} < D_2 k_t^{A_2} < D_2 k_t^{A_1}$ when $\kappa_0 < k_t < 1$,

$D_1 k_t^{A_1} < D_1 k_t^{A_2} < D_2 k_t^{A_1} < D_2 k_t^{A_2}$ when $1 < k_t < \kappa_1$,

$D_1 k_t^{A_1} < D_2 k_t^{A_1} < D_1 k_t^{A_2} < D_2 k_t^{A_2}$ when $0 < k_t < \kappa_0$.

Equation (4) thus leads to

$$E_Q(\delta k_t^a) = D_1 k_t^{A_2} (1-\theta^1_{22}) + D_2 k_t^{A_2} (\theta^1_{22}-\theta(A_1)) + D_1 k_t^{A_1} (\theta(A_1)-\theta_{12}) + D_2 k_t^{A_1} \theta_{12} \quad \text{when } 0 < k_t < \kappa_0,$$

$$E_Q(\delta k_t^a) = D_1 k_t^{A_2} (1-\theta^1_{22}) + D_1 k_t^{A_1} (\theta^1_{22}-\theta(\delta_2)) + D_2 k_t^{A_2} (\theta(\delta_2)-\theta_{12}) + D_2 k_t^{A_1} \theta_{12} \quad \text{when } \kappa_0 < k_t < 1,$$

$$E_Q(\delta k_t^a) = D_1 k_t^{A_1} (1-\theta^2_{12}) + D_1 k_t^{A_2} (\theta^2_{12}-\theta(\delta_2)) + D_2 k_t^{A_1} (\theta(\delta_2)-\theta_{22}) + D_2 k_t^{A_2} \theta_{22} \quad \text{when } 1 < k_t < \kappa_1,$$

$$E_Q(\delta k_t^a) = D_1 k_t^{A_1} (1-\theta^2_{12}) + D_2 k_t^{A_1} (\theta^2_{12}-\theta(A_2)) + D_1 k_t^{A_2} (\theta(A_2)-\theta_{22}) + D_2 k_t^{A_2} \theta_{22} \quad \text{when } 0 < k_t < \kappa_0.$$

where $\theta_{12} \equiv \theta(\{A_1, \delta_2\})$, $\theta_{22} \equiv \theta(\{A_2, \delta_2\})$, $\theta^1_{22} \equiv \theta(\{A_1, \delta_1\}\cup\{A_1, \delta_2\}\cup\{A_2, \delta_2\})$, and $\theta^2_{12} \equiv \theta(\{A_1, \delta_2\}\cup\{A_2, \delta_1\}\cup\{A_2, \delta_2\})$. Since $w_t = E_Q(\delta k_t^a) - k_t\, \partial E_Q(\delta k_t^a)/\partial k_t$, these equations lead to real wages in section 8.

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Table 1. The Range of $\lambda\delta$ for alternative values of A_1, A_2, and v

(a) v = 0.49

A_1	0.65	0.65	0.65	0.65	0.65
A_2	0.95	0.90	0.85	0.80	0.75
(1) Upper value	5.0761	4.4944	4.0323	3.6563	3.3445
(2) Lower value	4.9261	4.3956	3.9683	3.6166	3.3223

(b) v = 0.48

A_1	0.65	0.65	0.65	0.65	0.65
A_2	0.95	0.90	0.85	0.80	0.75
(1) Upper value	5.1546	4.5455	4.0650	3.6765	3.3557
(2) Lower value	4.8544	4.3478	3.9370	3.5971	3.3113

(c) v = 0.45

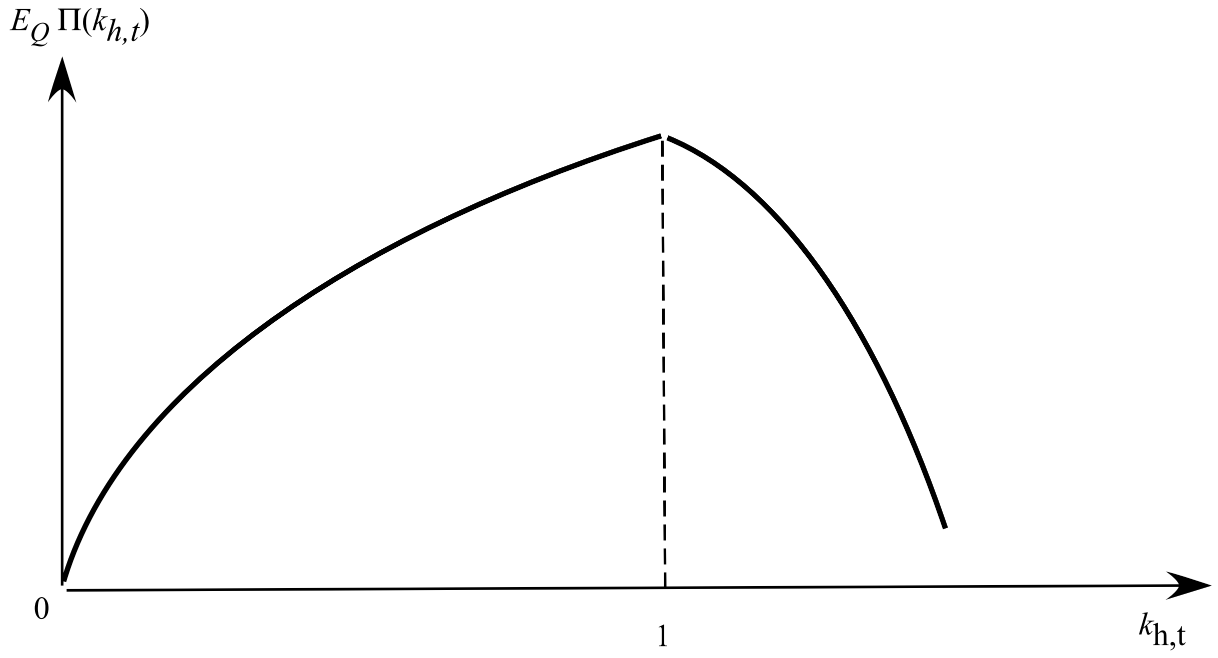
A_1	0.65	0.65	0.65	0.65	0.65
A_2	0.95	0.90	0.85	0.80	0.75
(1) Upper value	5.4054	4.7059	4.1667	3.7383	3.3898
(2) Lower value	4.6512	4.2105	3.8462	3.5398	3.2787

(d) v = 0.4

A_1	0.65	0.65	0.65	0.65	0.65
A_2	0.95	0.90	0.85	0.80	0.75
(1) Upper value	5.8824	5.0000	4.3478	3.8462	3.4483
(2) Lower value	4.3478	4.0000	3.7037	3.4483	3.2258

Figure 1. The Expected Profits under Knightian Uncertainy

(1) The case when $\delta E_Q a_{h,t} < R_t < -\delta E_Q(-a_{h,t})$.



(2) The case when $\delta E_Q a_{h,t} < -\delta E_Q(-a_{h,t}) < R_t$.

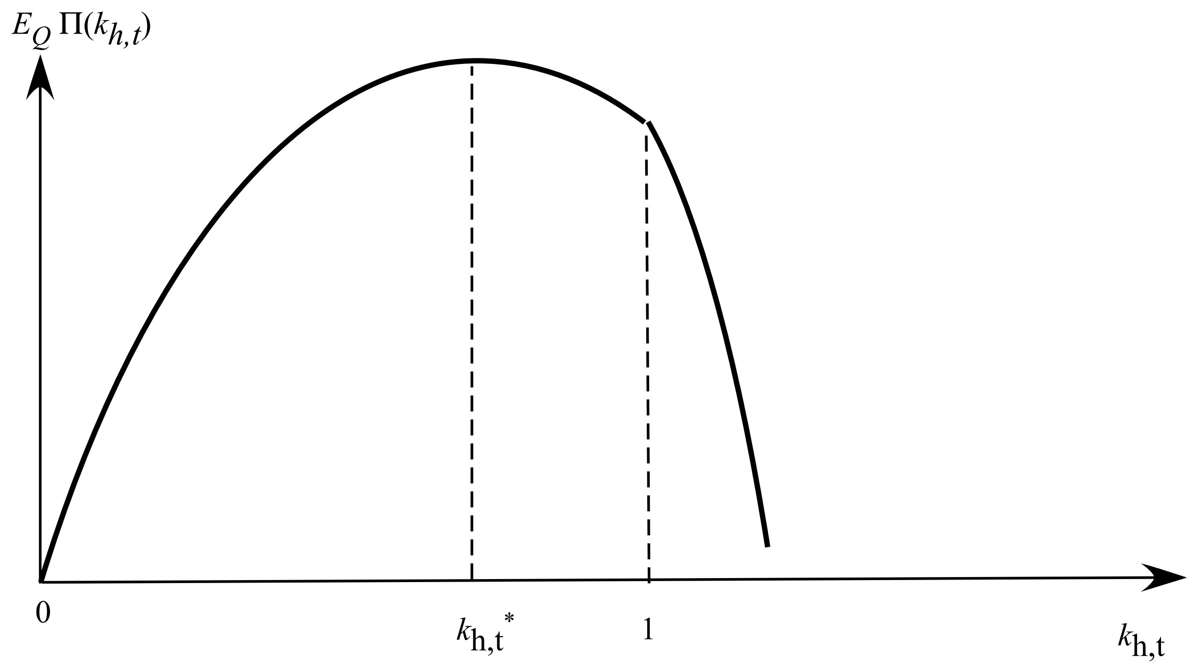


Figure 2. The Wage Function $w(k_t)$

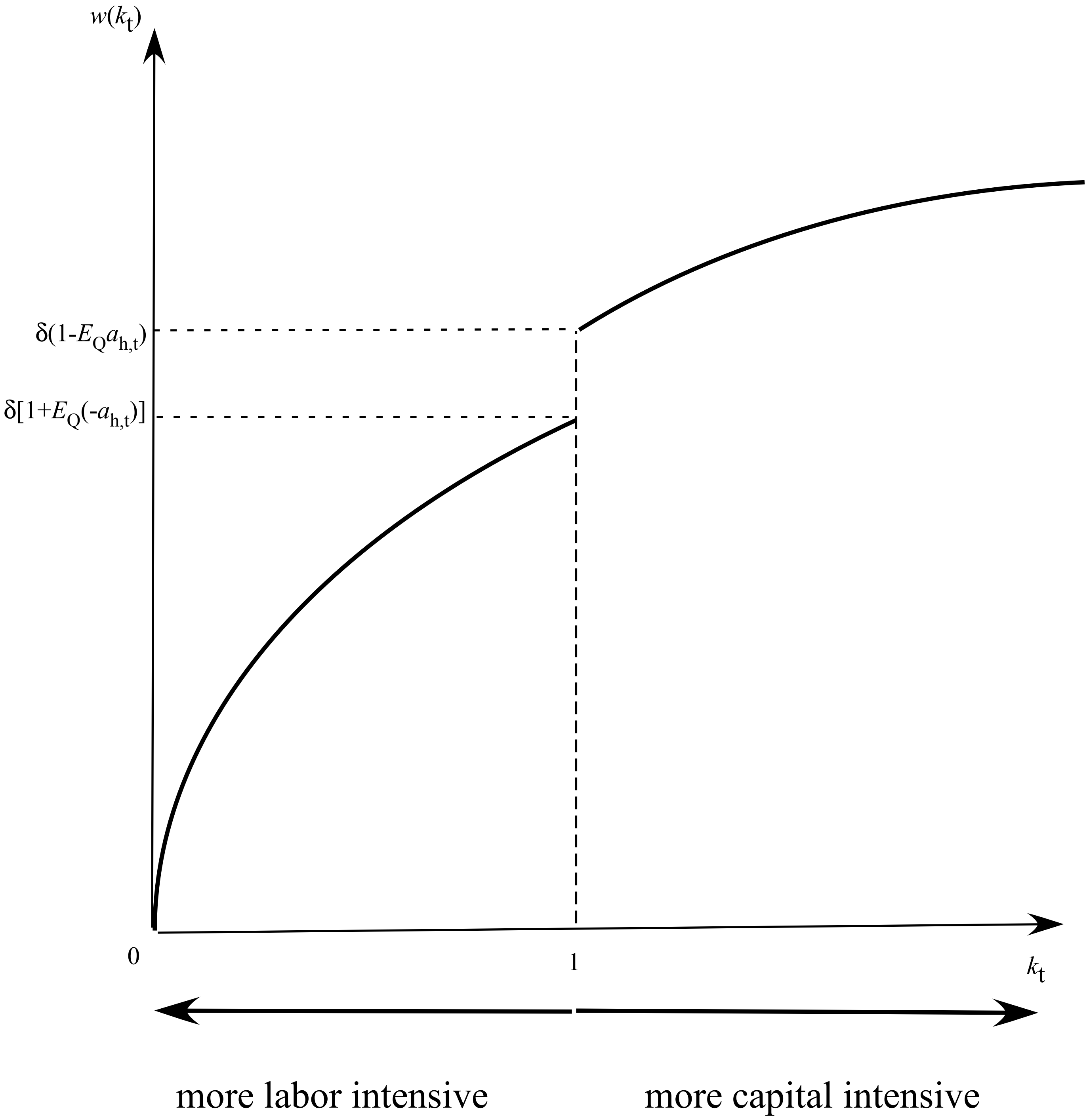


Figure 3-(I). The Dynamic Path of Capital Stock:
- The Case When $\lim_{k \to 1-0} w(k_t) > 1/\lambda$.

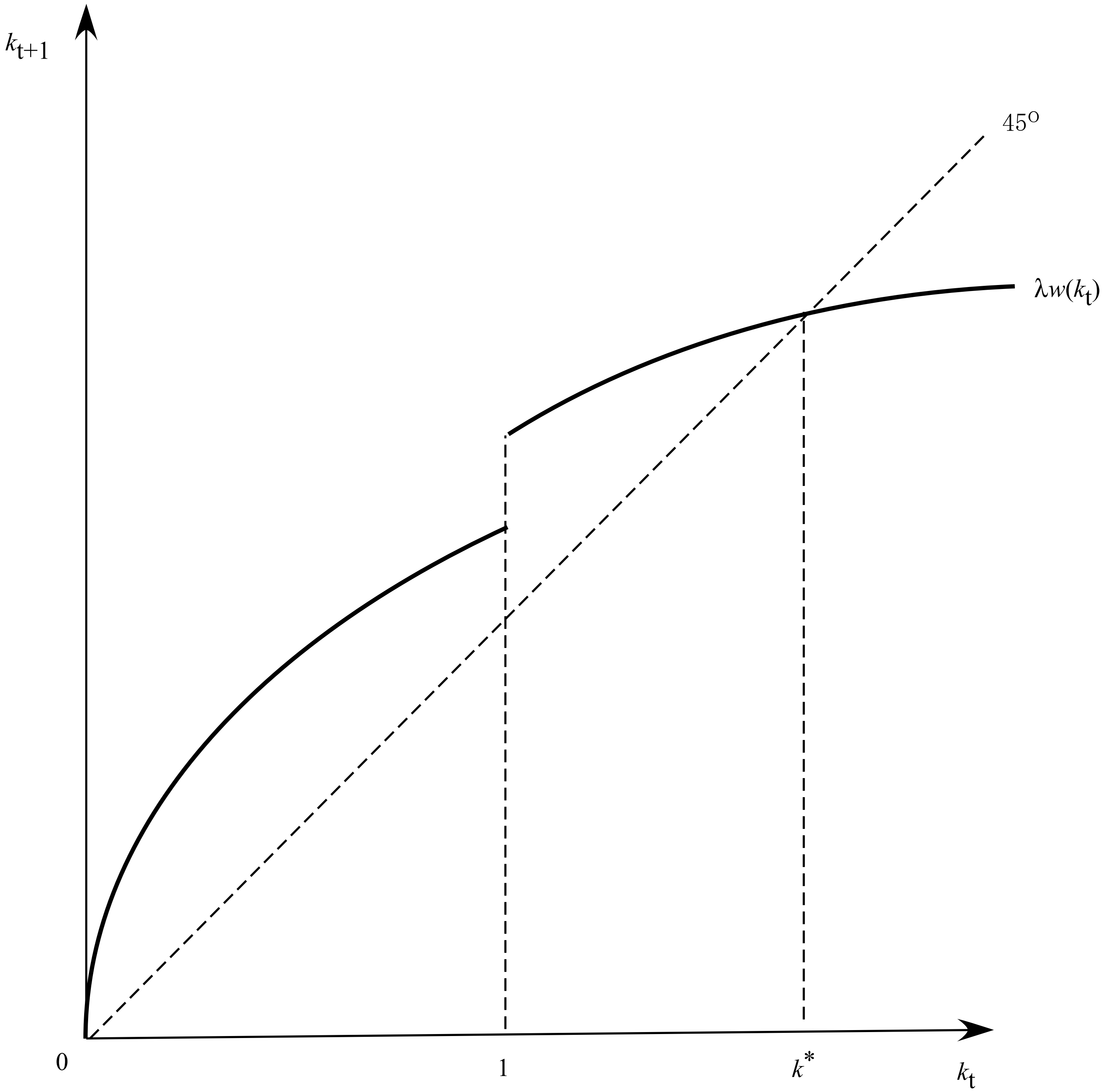


Figure 3-(II). The Dynamic Path of Capital Stock
- The Case When $\lim_{k\to 1+0} w(k_t) < 1/\lambda$.

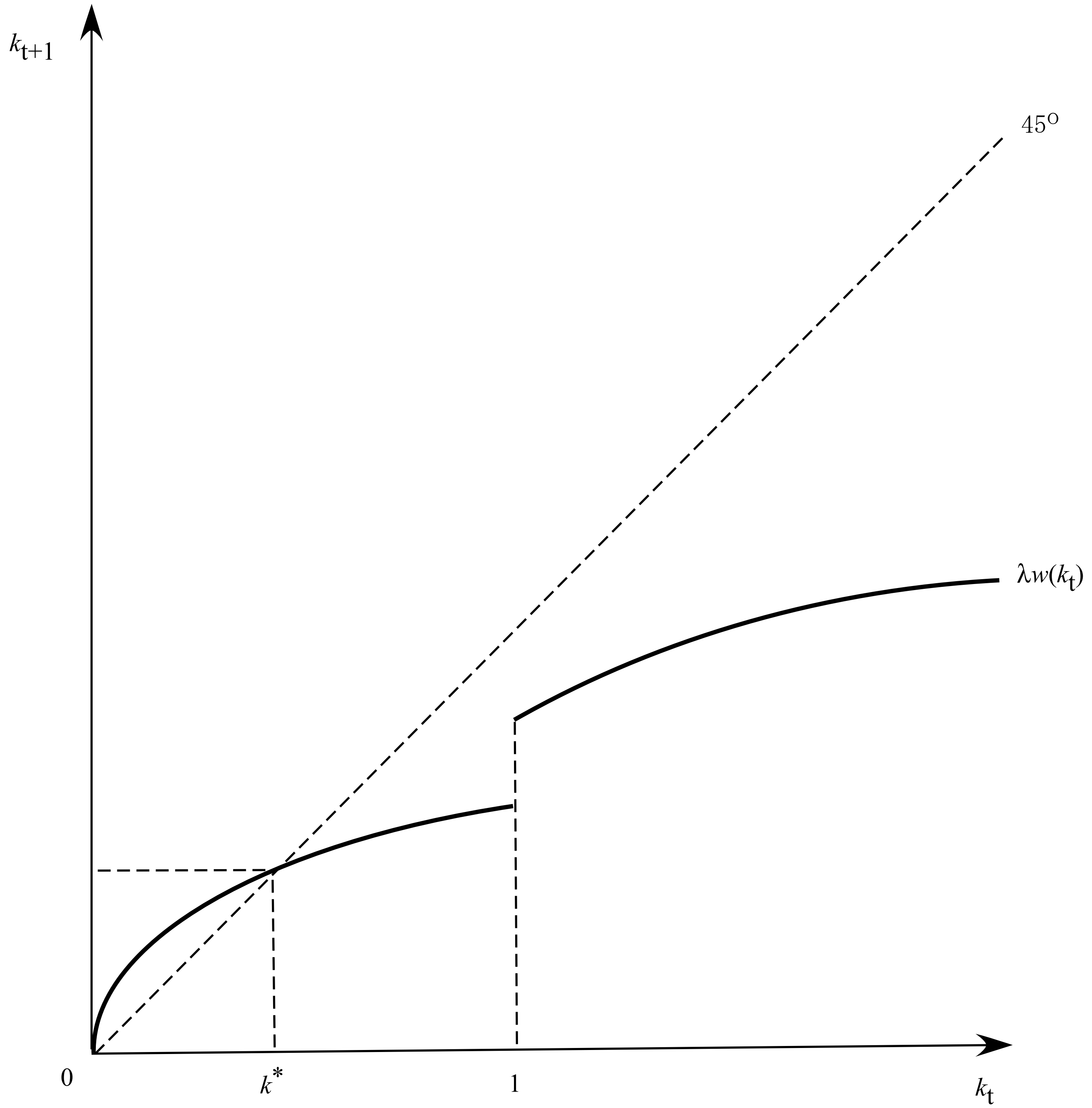


Figure 3-(III). The Dynamic Path of Capital Stock
- The Case When $\lim_{k \to 1-0} w(k_t) < 1/\lambda < \lim_{k \to 1+0} w(k_t)$.

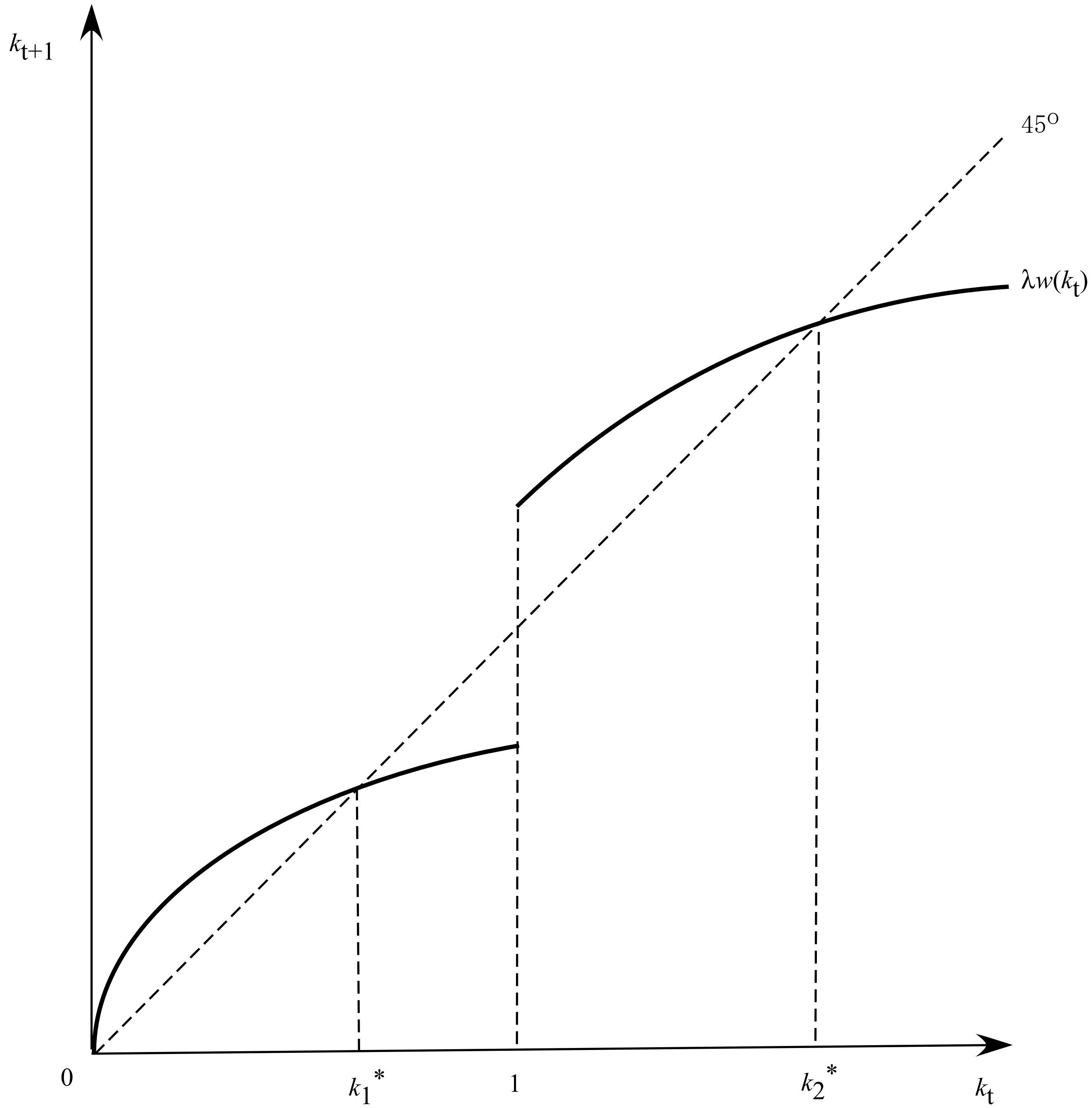


Figure 4. The Dynamic Path of Capital Stock
- Two Stable and One Unstable Non-zero Steady State Equilibria