Estimating Stochastic Volatility Models Using Daily Returns and Realized Volatility Simultaneously

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Abstract

Realized volatility, which is the sum of squared intraday returns over a certain interval such as a day, has recently attracted the attention of financial economists and econometricians as an accurate measure of the true volatility. In the real market, however, the presence of non-trading hours and market microstructure noise in transaction prices may cause the bias in the realized volatility. On the other hand, daily returns are less subject to the noise and therefore may provide additional information on the true volatility. From this point of view, we propose modeling realized volatility and daily returns simultaneously based on well-known stochastic volatility model. Using intraday data of Tokyo stock price index, we show that this model can estimate realized volatility biases and parameters simultaneously. We take a Bayesian approach and propose an efficient sampling algorithm to implement the Markov chain Monte Carlo method for our simultaneous model. The result of the model comparison between the simultaneous models using both naive and scaled realized volatilities indicates that the effect of non-trading hours is more essential than that of microstructure noise but still the latter has to be considered for better fitting. Our Bayesian approach has an advantage over the conventional two-step correction procedure in that we are able to take the uncertainty in estimation of both biases and parameters into account for the prediction and the evaluation of Value-at-Risk.

Key words: Bias correction; Markov chain Monte Carlo; Multi-move sampler; Realized volatility; Stochastic volatility

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1 Introduction

The financial return volatility, defined as the variance or the standard deviation of returns, plays a central role in the modern finance such as the option pricing and the evaluation of the Value-at-Risk (VaR). Realized volatility, which is the sum of squared intraday returns over a certain interval such as a day, has recently attracted the attention of financial economists and econometricians as an accurate measure of the true volatility. The realized volatility proposed by Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001) independently would provide a consistent estimator of the latent volatility in the ideal market assumption.

In the real market, however, there are two problems in measuring daily realized volatility from high frequency return data. One problem is the presence of non-trading hours and the other is the presence of the market microstructure noise in transaction prices. Stock markets are open only for a part of a day. For example, Tokyo Stock Exchange (TSE) is open only for 4.5 hours a day. The realized volatility may underestimate the latent one-day volatility if we define the latent one-day volatility for day t as the volatility from the market closing time for day $t-1$ to that for day t as usual and calculate the realized volatility as the sum of squared intraday returns only when the market is open. To avoid this underestimation, Hansen and Lunde (2005) scale realized volatility using daily returns so that the mean of the realized volatility equals to the variance of the daily return.

On the other hand, the market microstructure noise has various sources, including discrete trading and bid-ask spread (see e.g., O'Hara (1995) and Hasbrouck (2007) for details). Due to the noise, the realized volatility can be a biased estimator of the latent volatility (see, e.g., McAleer and Medeiros (2006) for a review of the realized volatility and effects of the microstructure noise). As the time interval approaches to zero, the variance of the true price process independent of the market microstructure noise decreases and then the effect of the microstructure noise becomes more significant. This means that there is a trade-off between the variance and bias of the realized volatility. Considering this trade-off, Bandi and Russell (2005) derive a simple formula to produce the optimal time interval of intraday returns used for calculating the realized volatility. Zhang, Mykland, and Aït-Sahalia (2005) also propose the way to correct the bias by combining two realized volatilities calculated from returns with different frequencies.

While the intraday returns are heavily contaminated by the microstructure noise, the daily returns are less subject to the noise. Thus the daily returns

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may provide an additional information on the latent volatility. From this perspective, this article models the daily returns and realized volatility simultaneously by extending stochastic volatility models.

We assume that the realized volatility includes the microstructure noise but still contains much information on the latent volatility. On the other hand, daily returns have less such noises but do not include the sufficient information on the latent volatility. Therefore, the model can estimate the biases due to both the microstructure noise and non-trading hours simultaneously without an additional calculation for determining the optimal time interval of Bandi and Russell (2005), subsampling of Zhang et al. (2005), or scaling of Hansen and Lunde (2005). Further, modeling returns and realized volatility simultaneously has a certain advantage over the two-step procedure by Giot and Laurent (2004) because the former enables us to estimate the distribution of returns, which is important in the prediction and the evaluation of the VaR, jointly with the biases and parameters in the volatility equation.

However, it is difficult to evaluate the likelihood of our model analytically and hence to estimate the parameters in the model by the maximum likelihood method. Thus we develop a Bayesian method for estimating the parameters in our model using the Markov chain Monte Carlo (MCMC) technique. To make the estimation method efficient, we extend the multi-move sampler proposed by Shephard and Pitt (1997) and Watanabe and Omori (2004). The MCMC method also enables us to take account of the parameter uncertainty in predicting the future volatility and the VaR.

We illustrate our model and estimation method by applying them to the daily data on returns and realized volatility of the Tokyo stock price index (TOPIX). We show that this model can estimate realized volatility biases and parameters simultaneously. Bayesian comparison between the simultaneous models using both two (naive and scaled) realized volatilities shows that the effect of non-trading hours is more essential than that of microstructure noise but still the latter has to be considered for better model fitting.

The paper is organized as follows. In Section 2, we first describe how to compute the realized volatility and discuss two practical problems in such computations. Then we propose a simultaneous model and explain its estimation method using the MCMC technique. Section 3 applies our proposed model to the TOPIX data. Finally, Section 4 concludes.

2 Simultaneous Modeling of Stochastic Volatility and Realized Volatility

2.1 Integrated Volatility, Realized Volatility, and Microstructure Noise

We first consider a simple continuous time process,

$$\mathrm{d}p(s) = \sigma(s)\mathrm{d}w(s), \tag{1}$$

where $p(s)$ denotes the log-price of a financial asset at time s, and $\sigma^2(s)$ is the instantaneous or spot volatility which is assumed to have locally square integrable sample paths and stochastically independent of the standard Brownian motion $w(s)$. Then, the volatility for day t is defined as the integral of $\sigma^2(s)$ over the interval $(t, t+1)$ where a full twenty-four-hour day is represented by the time interval 1, i.e.,

$$IV_t = \int_t^{t+1} \sigma^2(s)ds,$$

which is called an integrated volatility.

Although the integrated volatility cannot be observed, we can estimate it using observable high frequency asset returns. Suppose that we have n intraday returns during each day t, $\{r_{t,i}\}_{i=0}^{n-1}$, then the precise volatility measure, called a realized volatility, is defined as the squared sum of them over day t, i.e.,

$$RV_t = \sum_{i=0}^{n-1} r_{t,i}^2. \tag{2}$$

In the ideal world, that is, if there were no market microstructure noise and the asset were always and continuously traded, the realized volatility would provide a consistent estimator of the integrated volatility, that is,

$$RV_t \to IV_t, \quad n \to \infty.$$

Equivalently, the discretization noise due to $\mathrm{d}w(s)$ in the realized volatility disappears as the time interval goes to zero.

In the real market, however, there are two problems in measuring the realized volatility. One problem is the presence of non-trading hours and the other is the existence of the market microstructure noise.

Stock markets are open only during a part of a day. For example, Tokyo Stock Exchange (TSE) is open only for four and half hours a day. The realized volatility may underestimate the integrated volatility if we calculate the realized volatility as the sum of squared intraday returns only when the market is open. To avoid this underestimation, one may include returns on the non-trading hours (overnight and/or lunch time interval) but this can make the realized volatility noisy because such returns have much discretization noise. Thus, Hansen and Lunde (2005) propose scaling realized volatility for the market open period as

$$SRV_t = cRV_t, \quad c = \frac{\sum_{t=1}^{T}(R_t - \bar{R})^2}{\sum_{t=1}^{T} RV_t},$$

where R_t is the daily return, T is the daily sample size, and $\bar{R} = T^{-1}\sum_{t=1}^{T} R_t$. This ensures that the mean of the scaled realized volatility (SRV) is equal to the variance of daily returns.

On the other hand, to deal with the market microstructure noise, we denote the observed intraday log price as $p_{t,i}$ and suppose that the observed log price can be written as

$$p_{t,i} = p^*_{t,i} + \epsilon_{t,i},$$

where $p^*_{t,i}$ is the true intraday log price and $\epsilon_{t,i}$ is the microstructure noise. Then we can write the observed intraday return as the true intraday return $r^*_{t,i}$ plus the disturbance $\nu_{t,i} = \epsilon_{t,i} - \epsilon_{t,i-1}$, i.e.,

$$r_{t,i} = p_{t,i} - p_{t,i-1} = r^*_{t,i} + \nu_{t,i}. \tag{3}$$

Therefore, the realized volatility is given by

$$RV_t = \sum_{i=1}^{n}(r^*_{t,i})^2 + 2\sum_{i=1}^{n} r^*_{t,i}\nu_{t,i} + \sum_{i=1}^{n}\nu^2_{t,i}.$$

From this expression, we observe that the realized volatility can be a biased estimator of the integrated volatility. If the true price $p^*_{t,i}$ follows the equation (1), the mean of $\sum_{i=1}^{n}(r^*_{t,i})^2$ converges in probability to the integrated volatility as the time interval approaches to zero (equivalently, n goes to infinity). On the other hand, the expected value of $\sum_{i=1}^{n}\nu^2_{t,i}$ increases. For example, if $\epsilon_{t,i}$ has a constant variance σ^2_ϵ independent of the time interval and no autocorrelation, the expected value of $\sum_{i=1}^{n}\nu^2_{t,i}$ is equal to $2n\sigma^2_\epsilon$. This means that the bias caused by the microstructure noise increases as the time interval approaches to zero. Considering this trade-off between the variance and the bias of the realized volatility, Bandi and Russell (2005) derive a simple formula to produce the optimal time interval.

Bandi and Russell (2005) also show that $RV_t \to \infty$ in the case that $\epsilon_{t,i}$ has zero mean and is a covariance stationary stochastic process; the variance of $\nu_{t,i}$ is $O(1)$. Additionally, when the noise $\epsilon_{t,i}$ is an independent and identically distributed random variable and is independent of the price process, Zhang et al. (2005) show that RV_t has a bias and a larger variance due to the noise. Zhang et al. (2005) also propose a way to correct the bias by combining two realized volatilities calculated from returns with different frequencies. Furthermore, in the case of the dependent noise structure, Zhang (2006) and Aït-Sahalia, Mykland, and Zhang (2006) show that the similar result holds. See, e.g., McAleer and Medeiros (2006) for a review of the effects of the microstructure noise.

Suppose that $r^*_{t,i}$ and $\nu_{t,i}$ in the equation (3) are uncorrelated. Then, taking the variance of the both sides of the equation (3), we have

$$\mathrm{var}(r_{t,i}) = \mathrm{var}(r^*_{t,i}) + \mathrm{var}(\nu_{t,i})$$

If the true price $p^*_{t,i}$ follows the equation (1), $\mathrm{var}(r^*_{t,i})$ increases as the time interval increases. On the other hand, $\mathrm{var}(\nu_{t,i})$ remains the same ($2\sigma^2_\epsilon$ if $\epsilon_{t,i}$ has a constant variance σ^2_ϵ independent of the time interval and no autocorrelation). This means that the effect of the microstructure noise decreases as the time interval increases. Hence, daily returns are less subject to the microstructure noise than intraday returns. But daily returns suffer from another source of noise due to the discretization while the realized volatility is less subject to the discretization noise, which shows that daily returns and realized volatility can complement each other. This motivates us to model daily returns and realized volatility simultaneously as in the next subsection, which allows us to avoid additional calculations for adjusting the realized volatility.

2.2 Model

In this subsection, we propose a new model which utilizes daily returns and the realized volatility simultaneously. The model is an extension of the well-known stochastic volatility (SV) model (see for example Taylor (1986), Shephard (1996), and Ghysels, Harvey, and Renault (1996)). A simple SV model is written as,

$$\begin{aligned} R_t &= \exp(h_t/2)\epsilon_t, & \epsilon_t &\sim \mathrm{N}(0,1), & t &= 1,\ldots,T, \\ h_{t+1} &= \mu + \phi(h_t - \mu) + \eta_t, & \eta_t &\sim \mathrm{N}(0,\tau^2), & t &= 1,\ldots,T-1, \\ h_1 &= \mu + \eta_0, & \eta_0 &\sim \mathrm{N}\left(0, \frac{\tau^2}{1-\phi^2}\right), \end{aligned} \tag{4}$$

where h_t is the latent log volatility (log integrated volatility) at time t.

For notational convenience, let $y_{1,t}$ and $y_{2,t}$ denote a daily return and a logarithm of a realized volatility respectively. We extend the SV model as

$$\begin{aligned} y_{1,t} &= \exp(h_t/2)\epsilon_{1,t}, & \epsilon_{1,t} &\sim \mathrm{N}(0,1), \\ y_{2,t} &= h_t + \epsilon_{2,t}, & \epsilon_{2,t} &\sim \mathrm{N}(0,\sigma^2), \\ h_{t+1} &= \mu + \phi(h_t - \mu) + \eta_t, & \eta_t &\sim \mathrm{N}(0,\tau^2), \\ h_1 &= \mu + \eta_0, & \eta_0 &\sim \mathrm{N}\left(0, \frac{\tau^2}{1-\phi^2}\right), \end{aligned} \tag{5}$$

which we call the SV-RV model. Moreover, since the realized volatility can be biased due to the non-trading hours and microstructure noise, we modify the SV-RV model by adding the bias-correction term ξ in the second observation equation of (5), i.e.,

$$\begin{aligned} y_{1,t} &= \exp(h_t/2)\epsilon_{1,t}, & \epsilon_{1,t} &\sim \mathrm{N}(0,1), \\ y_{2,t} &= \xi + h_t + \epsilon_{2,t}, & \epsilon_{2,t} &\sim \mathrm{N}(0,\sigma^2), \\ h_{t+1} &= \mu + \phi(h_t - \mu) + \eta_t, & \eta_t &\sim \mathrm{N}(0,\tau^2), \\ h_1 &= \mu + \eta_0, & \eta_0 &\sim \mathrm{N}\left(0, \frac{\tau^2}{1-\phi^2}\right). \end{aligned} \tag{6}$$

If ξ is positive, realized volatility has an upward bias that may be due to the market microstructure noise and if ξ is negative, it has a downward bias due to the non-trading hours. Therefore, we may observe that the strength of effects of the microstructure noise and non-trading hours from the sign of ξ. We also call this model SV-RVC (SV-RV Corrected with respect to the bias due to the microstructure noise and non-trading hours) model.

The SV-RVC model can estimate the biases due to both the microstructure noise and non-trading hours simultaneously without the prior or two-step calculation for determining optimal time-interval of Bandi and Russell (2005), subsampling of Zhang et al. (2005), or scaling of Hansen and Lunde (2005). Further, SV-RV and SV-RVC models have a certain advantage over two-step procedures because the former enables us to estimate the distribution of returns, which is important in the evaluation of the VaR, jointly with the parameters in the volatility equation.

2.3 Markov Chain Monte Carlo Simulation

Because of the nonlinear relation between the daily return and the log latent volatility in equations (4), (5), and (6), we cannot compute the likelihood of these models by Kalman filter. But given $h = (h_1, \ldots, h_T)$, we can compute the conditional likelihood of the SV-RVC model as

$$f(y_{1,1}, y_{2,1}, \dots, y_{1,T}, y_{2,T}|\theta, h) \\ = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi}\exp(h_t/2)} \exp\left\{-\frac{y_{1,t}^2}{2\exp(h_t)}\right\} \times \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y_{2,t}-\xi-h_t)^2}{2\sigma^2}\right\},$$

where θ denotes the parameters. Hence, we take a Bayesian approach and estimate the posterior distribution of parameters in the SV, SV-RV, and SV-RVC models by considering h as additional latent variables. In this setup, efficient sampling h is the key to estimate the models. Therefore, we first describe the sampling algorithm for h.

2.3.1 Efficient Sampler for the latent volatilities

There are various sampling methods for h such as the single-move sampler proposed by Jacquier, Polson, and Rossi (1994) or the mixture sampler by Kim, Shephard, and Chib (1998). But the single-move sampler is extremely inefficient and the mixture sampler may not be able to be extended to more general models with risk premiums. Therefore, in this article, we use the multi-move sampler proposed by Shephard and Pitt (1997) and Watanabe and Omori (2004).

To illustrate the multi-move sampler, we consider the SV-RVC model. The observation equations of the model are

$$\begin{aligned} y_{1,t} &= \exp(h_t/2)\epsilon_{1,t}, & \epsilon_{1,t} &\sim \mathrm{N}(0,1), \\ y_{2,t} &= \xi + h_t + \epsilon_{2,t}, & \epsilon_{2,t} &\sim \mathrm{N}(0,\sigma^2), \end{aligned} \tag{7}$$

while state equations are written as

$$\begin{aligned} h_{t+1} &= (1-\phi)\mu + \phi h_t + \eta_t, & \eta_t &\sim \mathrm{N}(0,\tau^2), \\ h_1 &= \mu + \eta_0, & \eta_0 &\sim \mathrm{N}\left(0, \frac{\tau^2}{1-\phi^2}\right). \end{aligned} \tag{8}$$

To sample h from the posterior distribution, we divide $(h_1, \dots, h_T)$ into $K+1$ blocks, $(h_{k_{i-1}+1}, \dots, h_{k_i+1})'$, for $i = 0, \dots, K+1$, with $k_0 = 0$ and $k_{K+1} = T$. The selection of K knots, $(k_1, \dots, k_K)$, is implemented randomly and independently as

$$k_i = \mathrm{int}\{n \times (i + U_i)/(K+2)\}, \quad i = 1, \dots, K,$$

where U_i's are independent uniforms in $[0,1]$ and "int" denotes integer part. Since we sample each block given parameters $\theta \equiv (\xi, \sigma^2, \mu, \phi, \tau^2)$, other blocks, and observations $Y \equiv (y_1, \dots, y_T)$ where $y_t \equiv (y_{1,t}, y_{2,t})$, this sampling method is called a block sampler or multi-move sampler (e.g. Shephard and Pitt (1997)).

Suppose that $k_{i-1} = t-1$ and $k_i = t+k$. Then we sample $h^{(i)} \equiv (h_t, h_{t+1}, \ldots, h_{t+k})$ given $(h_1, \ldots, h_{t-1}, h_{t+k+1}, \ldots, h_T)$, θ, and Y. Since $h^{(i)}$ only depends on h_{t-1}, h_{t+k+1}, $(y_t, \ldots, y_{t+k})$ and θ, it is enough to consider sampling from the posterior distribution,

$$f(h^{(i)} | h_{t-1}, h_{t+k+1}, y_t, \ldots, y_{t+k}, \theta).$$

Given h_{t-1}, h_{t+k+1}, $(y_t, \ldots, y_{t+k})$ and θ, we can compute $h^{(i)}$ from $\eta^{(i)} \equiv (\eta_{t-1}, \ldots, \eta_{t+k+1})$ using equation (8). Thus, we consider sampling $\eta^{(i)}$ from the posterior distribution,

$$f(\eta^{(i)} | h_{t-1}, h_{t+k+1}, y_t, \ldots, y_{t+k}, \theta). \tag{9}$$

To construct a proposal distribution for the Metropolis-Hastings (MH) algorithm, we approximate this posterior density by the corresponding density of the linear Gaussian state space model (see Appendix A for details) given by

$$\begin{aligned} \hat{y}_{1,s} &= h_s + \hat{\epsilon}_{1,s}, & \hat{\epsilon}_{1,s} &\sim \mathrm{N}(0, v_s), \\ y_{2,s} &= \xi + h_s + \epsilon_{2,s}, & \epsilon_{2,s} &\sim \mathrm{N}(0, \sigma^2), \\ h_{s+1} &= \mu + \phi(h_s - \mu) + \eta_s, & \eta_s &\sim \mathrm{N}(0, \tau^2), \end{aligned} \tag{10}$$

where $\hat{y}_{1,s}$ and v_s are defined as,

(i) if $s = t, t+1, \ldots, t+k-1$ or $s = t+k = T$,

$$\hat{y}_{1,s} = \hat{h}_s + v_s l'(\hat{h}_s), \quad v_s = -\frac{1}{l''(\hat{h}_s)},$$

(ii) if $s = t+k < T$,

$$\begin{aligned} \hat{y}_s &= \hat{h}_s + v_s \left[l'(\hat{h}_s) + \frac{\phi}{\tau^2} \left\{ h_{t+k+1} - \mu - \phi(\hat{h}_s - \mu) \right\} \right], \\ v_s &= \frac{\tau^2}{\phi^2 - \tau^2 l''(\hat{h}_s)}, \end{aligned}$$

for some $\hat{h}_s$. We denote the posterior density of the $\eta^{(i)}$ from this linear Gaussian state space model by $g(\eta^{(i)})$.

We sample $\eta^{(i)}$ from the posterior density g using the simulation smoother of de Jong and Shephard (1995) and Durbin and Koopman (2002). But since g is the approximate density for the posterior density f, we use the acceptance-rejection Metropolis-Hasting (ARMH) algorithm proposed by Tierney (1994) (see also Chib and Greenberg (1995)) for sampling from f. We choose $\hat{h}_s$ as the posterior mode, which is calculated from the mode of $\eta^{(i)}$. In order to calculate the mode, $\hat{\eta}^{(i)}$, we first apply the disturbance smoother of Koopman (1993) with a starting value of $\hat{\eta}^{(i)}$. Then we apply the smoother for obtained $\hat{\eta}^{(i)}$ again. After some iterations, we can obtain the approximate mode of $\eta^{(i)}$ (see e.g. Shephard and Pitt (1997)).

2.3.2 Sampling Parameters

For the SV-RVC model in (6), we set priors as

$$
\begin{aligned}
&\xi \sim \mathrm{N}(m_\xi, s_\xi^2), \quad \sigma^2 \sim \mathrm{IG}\left(\frac{n_\sigma}{2}, \frac{d_\sigma}{2}\right), \\
&\mu \sim \mathrm{N}(m_\mu, s_\mu^2), \quad \frac{1+\phi}{2} \sim \mathrm{Beta}(a, b), \quad \tau^2 \sim \mathrm{IG}\left(\frac{n_\tau}{2}, \frac{d_\tau}{2}\right).
\end{aligned}
$$

Then, denoting $Y_1 = (y_{1,1}, \ldots, y_{1,T})$ and $Y_2 \equiv (y_{2,1}, \ldots, y_{2,T})$, the posterior density for $\theta \equiv (\xi, \sigma^2, \mu, \phi, \tau^2)$ and h becomes

$$
\begin{aligned}
&f(\theta, h|Y_1, Y_2) \\
&\propto \exp\left[-\frac{1}{2}\sum_{t=1}^{T}\left\{h_t - y_{1,t}^2 \exp(-h_t)\right\}\right] \\
&\quad \times (\sigma^2)^{-T/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{T}(y_{2,t} - \xi - h_t)^2\right\} \\
&\quad \times \sqrt{1-\phi^2}(\tau^2)^{-T/2} \\
&\quad \times \exp\left\{-\frac{1}{2\tau^2}(1-\phi^2)(h_1-\mu)^2 - \sum_{t=1}^{T-1}(h_{t+1} - (1-\phi)\mu - \phi h_t)^2\right\} \\
&\quad \times \exp\left\{-\frac{(\xi - m_\xi)^2}{2s_\xi^2}\right\} \times (\sigma^2)^{-(n_\sigma/2+1)} \exp\left(-\frac{d_\sigma}{2\sigma^2}\right) \\
&\quad \times \exp\left\{-\frac{(\mu - m_\mu)^2}{2s_\mu^2}\right\} \times \left(\frac{1+\phi}{2}\right)^{a-1}\left(\frac{1-\phi}{2}\right)^{b-1} \\
&\quad \times (\tau^2)^{-(n_\tau/2+1)} \exp\left(-\frac{d_\tau}{2\tau^2}\right).
\end{aligned}
$$

To implement the Markov chain Monte Carlo simulation, we sample from the posterior distribution as follows:

Step 1. Simulate h from $f(h|\mu, \phi, \tau^2, Y_1, Y_2)$.
Step 2. Simulate ξ from $f(\xi|\sigma^2, h, Y_2)$.
Step 3. Simulate σ^2 from $f(\sigma^2|\xi, h, Y_2)$.
Step 4. Simulate μ from $f(\mu|\phi, \tau^2, h)$.
Step 5. Simulate τ^2 from $f(\tau^2|\mu, \phi, h)$.
Step 6. Simulate ϕ from $f(\phi|\mu, \tau^2, h)$.

We note that ξ and σ^2 only depend on the observation equation (7) given h while μ, ϕ, and τ^2 only depend on the volatility equation (8) given h.

In the first step, we conduct the multi-move sampler described in the previous subsection. In the second and third steps, we sample from the conditional posterior distributions of ξ and τ^2,

$$\xi|\sigma^2, Y_2, h \sim \mathrm{N}(\tilde{m}_\xi, \tilde{s}_\xi^2), \quad \sigma^2|\xi, Y_2, h \sim \mathrm{IG}\left(\frac{\tilde{n}_\sigma}{2}, \frac{\tilde{d}_\sigma}{2}\right),$$

where

$$\begin{aligned}
\tilde{m}_\xi &= \frac{s_\xi^2(y_{2,t} - h_t) + \sigma^2 m_\xi}{T s_\xi^2 + \sigma^2}, \quad \tilde{s}_\xi^2 = \frac{s_\xi^2 \sigma^2}{T s_\xi^2 + \sigma^2}, \\
\tilde{n}_\sigma &= T + n_\sigma, \quad \tilde{d}_\sigma = d_\sigma + \sum_{t=1}^{T} (y_{2,t} - \xi - h_t)^2.
\end{aligned}$$

In the fourth and fifth steps, we generate samples from the conditional posterior distributions of μ and τ^2

$$\mu|\phi, \tau^2, h \sim \mathrm{N}(\tilde{m}_\mu, \tilde{s}_\mu^2), \quad \tau^2|\mu, \phi, h \sim \mathrm{IG}\left(\frac{\tilde{n}_\tau}{2}, \frac{\tilde{d}_\tau}{2}\right),$$

where

$$\begin{aligned}
\tilde{m}_\mu &= \tilde{s}_\mu^2 \left\{ \frac{(1-\phi^2)}{\tau^2} h_1 + \frac{(1-\phi)}{\tau^2} \sum_{t=1}^{T-1} (h_{t+1} - \phi h_t) + \frac{m_\mu}{s_\mu^2} \right\}, \\
\tilde{s}_\mu^2 &= \frac{s_\mu^2 \tau^2}{s_\mu^2 \{(T-1)(1-\phi)^2 + 1 - \phi^2\} + \tau^2}, \\
\tilde{n}_\tau &= T + n_\tau, \\
\tilde{d}_\tau &= d_\tau + (h_1 - \mu)^2 (1 - \phi^2) + \sum_{t=1}^{T-1} \{h_{t+1} - \mu - \phi(h_t - \mu)\}^2.
\end{aligned}$$

In the final step, the logarithm of the posterior density is

$$\begin{aligned}
&\log f(\phi|\mu, \tau^2, h_1, \ldots, h_T) \\
&= \text{const.} + \log\{\varphi(\phi)\} - \frac{1}{2\tau^2} \sum_{t=1}^{T-1} \{h_{t+1} - \mu - \phi(h_t - \mu)\}^2,
\end{aligned}$$

where $|\phi| < 1$ and

$$\begin{aligned}
\log \varphi(\phi) = &(a-1)\log(1+\phi) + (b-1)\log(1-\phi) \\
&- \frac{(h_1 - \mu)^2 (1 - \phi^2)}{2\tau^2} + \frac{1}{2}\log(1 - \phi^2).
\end{aligned}$$

In order to sample from this density, we employ Metropolis-Hastings (MH) algorithm (Chib and Greenberg (1995)). We construct the proposal density $h(\phi)$ as an approximation to the conditional posterior density by omitting the term $\log \varphi(\phi)$, i.e.,

$$\begin{aligned}
\log f(\phi|\mu, \tau^2, h_1, \ldots, h_T) &\approx \text{const.} - \frac{1}{2\tau^2}\sum_{t=1}^{T-1}\{h_{t+1} - \mu - \phi(h_t - \mu)\}^2 \\
&= \text{const.} - \frac{(\phi - m_\phi)^2}{2s_\phi^2} \\
&= \text{const.} + \log h(\phi),
\end{aligned}$$

where

$$m_\phi = \frac{\sum_{t=1}^{T-1}(h_{t+1} - \mu)(h_t - \mu)}{\sum_{t=1}^{T-1}(h_t - \mu)^2}, \quad s_\phi^2 = \frac{\tau^2}{\sum_{t=1}^{T-1}(h_t - \mu)^2}.$$

If ϕ_{j-1} is a current sample of ϕ, we propose a candidate ϕ^p for ϕ_j by sampling from $\mathrm{N}(m_\phi, s_\phi^2)$ truncated on $(-1, 1)$ and accepting it with probability

$$q = \min\left\{\frac{f(\phi^p)h(\phi_{j-1})}{f(\phi_{j-1})h(\phi^p)}, 1\right\} = \min\left\{\frac{\varphi(\phi^p)}{\varphi(\phi_{j-1})}, 1\right\}.$$

3 Application to Stock Return Data

3.1 Data and Realized Volatility

We use the high frequency data of Tokyo stock price index (TOPIX) obtained from the Nikkei NEEDS MT tick data. The price is preserved in one minute frequency during the period from April 1, 1996 to March 31, 2005 (2216 trading days). TSE is open for 9:00-11:00 (morning session) and 12:30-15:00 (afternoon session) in usual trading days and only for 9:00-11:00 in the first and last trading days in every year. Excluding the overnight and lunch time intervals, we obtain 119 intraday returns in the morning session and 149 returns in the afternoon session.

To confirm that SV-RVC model can correct the bias due to market microstructure noise and non-trading hours, we use the realized volatilities calculated from 1-, 5-, and 10-minute intraday returns when the market is open. We compute RV_t^m by omitting the overnight and lunch time interval return (we also compute RV_t^m in the first and last trading days in every year using only morning session returns), where $m = 1, 5, 10$ denote the time interval used for calculating realized volatilities. Following Hansen and Lunde (2005), we also calculate scaled realized volatility as,

$$SRV_t^m = cRV_t^m, \quad c = \frac{\sum_{t=1}^T (R_t - \bar{R})^2}{\sum_{t=1}^T RV_t^m}.$$

Values of c are 3.6711, 2.9645, and 2.7881 for SRV_t^1, SRV_t^5, and SRV_t^{10} respectively. Since all these values are smaller than $24/4.5 = 5.3333$, we confirm that non-trading hours contribute to the increase in volatility less than trading hours, which is consistent with previous findings (Fama (1965), French and Roll (1986), and Nelson (1991)).

Figures 1 - 3 are the realized volatility calculated using intraday returns only when the market is open (RV^m), scaled one (SRV^m), and their logarithms ($\log(RV^m)$, $\log(SRV^m)$) at the time interval $m = 1, 5, 10$ (minute). Figure 4 plots daily return (R), its absolute value ($|R|$), squared return (R^2), and its logarithm ($\log R^2$). They show that the variation of realized volatilities is smaller than that of squared daily return, which is due to the discretization noise in daily returns.

Table 1 shows descriptive statistics. We observe four interesting results from this table. First, the mean of RV^m ($m = 1, 5, 10$) is smaller than that of the squared daily return, which implies there is a negative bias in the realized volatility due to non-trading hours. The standard deviation of the realized volatilities is much smaller than the squared return as we expected from Figures 1 - 4. Second, the standard deviation of RV^m becomes larger as the time interval m increases, which confirms the intraday return becomes noisy due to the discretization effect as the interval increases. These results suggest that the more precise estimate of the true volatility may be obtained by correcting the bias due to non-trading hours and microstructure noise in RV^1.

Third, the skewness and kurtosis indicate that the realized volatilities are not Gaussian but their logarithms are nearly Gaussian, which motivates us to model the logarithm of realized volatilities instead of the realized volatilities. Finally, LB(10), the heteroskedasticity-corrected Ljung-Box statistic including 10 lags calculated following Diebold (1988), shows that daily return is not autocorrelated while volatilities, especially the log realized volatilities, are autocorrelated significantly at the one percent level. This result is consistent with the well-known phenomenon of volatility clustering.

These findings are in accordance with previous studies: Andersen, Bollerslev, Diebold, and Labys (2001b) for exchange rates; Andersen, Bollerslev, Diebold, and Ebens (2001a) for stocks; Martens (2002) for stock index futures; and Watanabe and Yamaguchi (2007) for Japanese stock index (Nikkei 225).

3.2 Estimation Results

Tables 3 - 6 summarize MCMC estimation results of SV, SV-RV, and SV-RVC models obtained by 5000 samples recorded after discarding 1000 samples from MCMC iterations (all calculations in this paper are done by using Ox (Doornik (2002))). We apply the latter two models to both realized volatilities for the market open period (RV^m) and scaled one (SRV^m) at each time interval ($m = 1, 5, 10$) and denote models using scaled realized volatilities as SV-SRV and SV-SRVC models. CD is the p-value of the convergence diagnostic (CD) test by Geweke (1992). The inefficiency factor is defined as $1 + 2\sum_{s=1}^{\infty} \rho_s$ where ρ_s is the sample autocorrelation at lag s, and are computed to measure how well the MCMC chain mixes (see e.g. Chib (2001)). It is the ratio of the numerical variance of the posterior sample mean to the variance of the posterior sample mean from uncorrelated draws. The inverse of inefficiency factor is also known as relative numerical efficiency (Geweke (1992)). When the inefficiency factor is equal to m, we need to draw MCMC samples m times as many as uncorrelated samples.

Table 3 shows that ϕ is estimated relatively lower in the SV-RV models using RV^m although ϕ is expected to be close to one as a result of the strong autocorrelations of log realized volatilities. Since ϕ is close to one in the SV-RVC, SV-SRV, and SV-SRVC models in Tables 4, 5, and 6, this is probably because the bias is not corrected appropriately for the non-trading hours.

In Table 4, the posterior means of ξ in the SV-RVC models are all negative. This implies that the effect of non-trading hours is stronger than that of microstructure noise. We note that the posterior mean of $\xi = -1.2344$ in the SV-RVC model using RV^1 is larger than the scaling factor, $-\log(c) = -1.3005$, and, further, the posterior probability that ξ is positive is greater than 0.95 in the SV-SRVC model using SRV^1 in Table 6. From these results, we observe that the bias due to the microstructure noise still exists even after scaling, which means that correcting the bias due to non-trading hours is not sufficient for adjusting the total bias in the realized volatility. We also note that the difference between -1.2344 and -1.3005 is 0.0661 which is almost equal to the posterior mean of $\xi = 0.0655$ in SV-SRVC model in Table 6.

Table 6 shows that 95% credible intervals of ξ contain zeros for the SV-SRVC models using SRV^5 and SRV^{10}. This result shows that the bias due to the microstructure noise disappears as the time interval m increases. On the contrary, the variances of realized volatility (σ^2) and latent log volatility (τ^2) increase as m increases. This bias-variance trade-off is consistent with previous studies such as Bandi and Russell (2005) and Hansen and Lunde (2006). While these research suggest taking the optimal time interval for dealing with this trade-off, our SV-RVC model can correct the bias without considering a selection of such a time interval. Therefore, the model provides volatility estimator with the least variation by using the realized volatility calculated from intraday returns with the shortest interval (one minute). Moreover, the model allows us to skip the two-step procedure determining the optimal sampling frequency for calculating the realized volatility and bias-correcting procedures, both of which need complicated calculations.

To investigate the effect of lunch time non-trading hours, we also estimate the models using realized volatilities and scaled ones calculated from intraday returns including the lunch time interval. The results are the same as those when we used RV^m ($m = 1, 5, 10$) and hence are omitted.

3.3 Model Comparisons Using Marginal Likelihoods

For model comparisons, we calculate marginal likelihoods of these models. We follow Chib (1995) and Chib and Jeliazkov (2001) to calculate the posterior ordinate and its numerical standard error. The likelihood ordinate is computed by using the auxiliary particle filter of Pitt and Shephard (1999). We calculate the estimate of the likelihood ordinate and its standard error as the sample mean and standard deviation of the likelihoods from 20 iterations. Table 7 shows the logarithm of marginal likelihoods (standard errors are in the parentheses).

From this table, we can confirm that the bias-correction is essential for model fitting. Especially, correcting the bias due to non-trading hours gives more significant improvement (see SV-RV and SV-RVC models) than adjusting it due to the microstructure noise (see SV-SRV and SV-SRVC models). Comparing the log marginal likelihoods of SV-SRV and SV-SRVC models using SRV^1, however, we observe that the bias due to microstructure noise still exists and the SV-SRVC model can adequately correct the bias. But the log marginal likelihoods of both models using SRV^5 and SRV^{10} show the benefit for correcting the bias due to microstructure noise disappears as the time interval m increases. This result is consistent with the previous studies suggesting the use of intraday returns with longer intervals such as 5 or 10 minutes to calculate the realized volatility with less microstructure noise (see e.g. Bandi and Russell (2005) and Hansen and Lunde (2006)).

4 Concluding Remarks

In this paper, we proposed modeling daily returns and realized volatility simultaneously extending the well-known stochastic volatility model and described the efficient sampling algorithm for our model to implement Markov chain Monte Carlo simulation. We show that this model can jointly estimate the parameters and the realized volatility bias due to both non-trading hours and the market microstructure noise. Especially, this model allows us to skip determining the optimal sampling frequency for calculating the realized volatility and bias-correcting procedures, both of which need complicated calculations. Comparison of marginal likelihood between the simultaneous models using both naive and scaled realized volatilities shows that the effect of non-trading hours is more essential than that of microstructure noise but still the latter has to be considered for better model fitting.

Using Bayesian approach, our model can consider the uncertainty in the estimation of the biases and parameters when we derive the predictive distribution of daily returns, which is important for the evaluation of the VaR estimation. The comparison of the forecasting performances using the VaR for various models such as the ARFIMA model would be our future work. Further, although we use only the standard normal distribution for daily returns in this paper, our model can be applied to other distributions for daily returns such as Student's t, skewed-t, and normal inverse Gaussian (NIG) distributions. Especially, the NIG distribution has recently attracted the attention of financial economists and econometricians since conditional distribution of the returns is distributed as NIG if the realized volatility is conditionally inverse Gaussian and daily return standardized by the realized volatility is approximately Gaussian (see e.g., Forsberg (2002) and Forsberg and Bollerslev (2002)).

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Appendix

A Approximation of the conditional posterior density of $\eta^{(i)}$

In the case of $t+k<T$, the log of posterior density (9) is written as

$$
\begin{aligned}
&\log f(\eta^{(i)}|h_{t-1}, h_{t+k+1}, y_t, \dots, y_{t+k}, \theta) \\
&= \text{const.} + \log f(y_{1,t}, \dots, y_{1,t+k}|h_t, \dots, h_{t+k}) \\
&\qquad + \log f(y_{2,t}, \dots, y_{2,t+k}|\xi, \sigma^2, h_t, \dots, h_{t+k}) \\
&\qquad + \log f(h_{t+k+1}|\mu, \phi, \tau^2, h_{t+k}) + \log f(\eta_{t-1}, \dots, \eta_{t+k-1}|\tau^2) \\
&= \text{const.} - \sum_{s=t}^{t+k} \left\{ \frac{h_s}{2} + \frac{y_{1,s}^2}{2} \exp(-h_s) \right\} - \frac{1}{2\sigma^2} \sum_{s=t}^{t+k} (y_{2,s} - \xi - h_s)^2 \\
&\qquad - \frac{1}{2\tau^2} \{h_{t+k+1} - \mu - \phi(h_{t+k} - \mu)\}^2 - \frac{1}{2\tau^2} \sum_{s=t-1}^{t+k-1} \eta_s^2. \qquad \text{(A.1)}
\end{aligned}
$$

Following Shephard and Pitt (1997), we approximate this log-posterior density by Taylor expansion of

$$
l(h_s) \equiv -\frac{h_s}{2} - \frac{y_{1,s}^2}{2} \exp(-h_s)
$$

around $h_s = \hat{h}_s$ as follows;

$$
\begin{aligned}
&\log f(\eta^{(i)}|h_{t-1}, h_{t+k+1}, y_t, \dots, y_{t+k}, \theta) \\
&\approx \text{const.} + \sum_{s=t}^{t+k} \left\{ l(\hat{h}_s) + (h_s - \hat{h}_s) l'(\hat{h}_s) + \frac{1}{2}(h_s - \hat{h}_s)^2 l''(\hat{h}_s) \right\} \\
&\qquad - \frac{1}{2\sigma^2} \sum_{s=t}^{t+k} (y_{2,s} - \xi - h_s)^2 \\
&\qquad - \frac{1}{2\tau^2} \{h_{t+k+1} - \mu - \phi(h_{t+k} - \mu)\}^2 - \frac{1}{2\tau^2} \sum_{s=t-1}^{t+k-1} \eta_s^2 \\
&\equiv \log cg(\eta_{t-1}, \dots, \eta_{t+k-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
l'(\hat{h}_s) &\equiv \frac{\partial l(\hat{h}_s)}{\partial h_s} = \frac{1}{2} \left\{ y_s^2 \exp(-\hat{h}_s - 1 \right\}, \\
l''(\hat{h}_s) &\equiv \frac{\partial^2 l(\hat{h}_s)}{\partial {h_s}^2} = -\frac{y_s^2}{2} \exp(-\hat{h}_s).
\end{aligned}
$$

On the other hand, when sampling the last block, i.e. $t+k=T$, the log of posterior density is written as, excluding h_{t+k+1} in the condition,

$$
\begin{aligned}
&\log f(\eta^{(i)}|h_{t-1}, y_t, \dots, y_{t+k}, \theta) \\
&= \text{const.} - \sum_{s=t}^{t+k} \left\{ \frac{h_s}{2} + \frac{y_{1,s}^2}{2} \exp(-h_s) \right\} - \frac{1}{2\sigma^2} \sum_{s=t}^{t+k} (y_{2,s} - \xi - h_s)^2 \\
&\quad - \frac{1}{2\tau^2} \sum_{s=t-1}^{t+k-1} \eta_s^2.
\end{aligned}
$$

Similar to the case of $t + k < T$, we approximate this log-density as

$$
\begin{aligned}
&\log f(\eta^{(i)}|h_{t-1}, y_t, \dots, y_{t+k}, \theta) \\
&\approx \text{const.} + \sum_{s=t}^{t+k} \left\{ l(\hat{h}_s) + (h_s - \hat{h}_s) l'(\hat{h}_s) + \frac{1}{2}(h_s - \hat{h}_s)^2 l''(\hat{h}_s) \right\} \\
&\quad - \frac{1}{2\sigma^2} \sum_{s=t}^{t+k} (y_{2,s} - \xi - h_s)^2 - \frac{1}{2\tau^2} \sum_{s=t-1}^{t+k-1} \eta_s^2 \\
&\equiv \log cg(\eta_{t-1}, \dots, \eta_{t+k-1}).
\end{aligned}
$$

Then we can consider $g(\eta_{t-1}, \dots, \eta_{t+k-1})$ as the conditional density of linear Gaussian state space model,

$$
\begin{aligned}
\hat{y}_{1,s} &= h_s + \hat{\epsilon}_{1,s}, & \hat{\epsilon}_{1,s} &\sim \mathrm{N}(0, v_s), \\
y_{2,s} &= \xi + h_s + \epsilon_{2,s}, & \epsilon_{2,s} &\sim \mathrm{N}(0, \sigma^2), \\
h_{s+1} &= \mu + \phi(h_s - \mu) + \eta_s, & \eta_s &\sim \mathrm{N}(0, \tau^2),
\end{aligned}
$$

where $\hat{y}_{1,s}$ and v_s are defined as,

(i) if $s = t, t+1, \dots, t+k-1$ or $s = t + k = T$,

$$
\hat{y}_{1,s} = \hat{h}_s + v_s l'(\hat{h}_s), \quad v_s = -\frac{1}{l''(\hat{h}_s)},
$$

(ii) if $s = t + k < T$,

$$
\begin{aligned}
\hat{y}_s &= \hat{h}_s + v_s \left[l'(\hat{h}_s) + \frac{\phi}{\tau^2} \left\{ h_{t+k+1} - \mu - \phi(\hat{h}_s - \mu) \right\} \right], \\
v_s &= \frac{\tau^2}{\phi^2 - \tau^2 l''(\hat{h}_s)}.
\end{aligned}
$$

The correction in (ii) is necessary except the last block ($s = t + k = T$) because of the existence of the fourth term in equation (A.1) (see Watanabe and Omori (2004)).

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Table 1
Descriptive statistics for realized volatilities for the market open period (RV^m), scaled realized volatility (SRV^m), log of them at frequency $m = 1, 5, 10$ (minute), daily return (R), its absolute value ($|R|$), squared return (R^2), and log of them ($\log |R|$, $\log R^2$) during the period from 1 April 1996 to 31 March 2005 (2216 trading days). LB(10) is the heteroskedasticity-corrected Ljung-Box statistics of Diebold (1988) with 10 lags. The critical values for LB_{10} are: 15.99 (10%), 18.31 (5%), and 23.21 (1%).

	RV^1	RV^5	RV^{10}	SRV^1	SRV^5	SRV^{10}
			Level			
Mean	0.4424	0.5478	0.5825	1.6240	1.6240	1.6240
Stdev	0.3089	0.4923	0.5722	1.1340	1.4595	1.5955
Skewness	4.6566	4.3165	4.7976	4.6566	4.3165	4.7976
Kurtosis	64.2218	38.3029	42.8742	64.2218	38.3029	42.8742
Max	6.2472	6.7510	7.7187	22.9340	20.0133	21.5208
Min	0.0665	0.0367	0.0332	0.2440	0.1087	0.0927
LB(10)	1294.09	1095.32	751.79	1294.09	1095.32	751.79
			Logarithm			
Mean	−0.9940	−0.8739	−0.8395	0.3065	0.2128	0.1859
Stdev	0.5937	0.7350	0.7643	0.5937	0.7350	0.7643
Skewness	0.0157	−0.0256	0.0250	0.0157	−0.0256	0.0250
Kurtosis	3.1601	3.2047	3.2493	3.1601	3.2047	3.2493
Max	1.8321	1.9097	2.0436	3.1326	2.9964	3.0690
Min	−2.7111	−3.3061	−3.4043	−1.4106	−2.2194	−2.3789
LB(10)	4875.28	4044.76	3359.10	4875.28	4044.76	3359.10
	R	$\|R\|$	R^2	$\log \|R\|$	$\log R^2$	
Mean	−0.0147	0.9573	1.6242	−0.5046	−1.0091	
Stdev	1.2744	0.8413	3.2096	1.1353	2.2707	
Skewness	−0.1084	1.7813	6.0007	−1.2009	−1.2009	
Kurtosis	4.9005	8.2747	57.9221	5.6049	5.6049	
Max	6.5993	6.5993	43.5513	1.8870	3.7739	
Min	−6.5736	0.0006	0.0000	−7.3682	−14.7364	
LB(10)	20.42	189.96	100.81	73.45	73.45	

Table 2
Estimation results of SV model. The last two columns are p-value of Geweke's convergence diagnostic (CD) test and inefficiency factor, respectively.

		Mean	Stdev	95% interval	CD	Inef.
SV	μ	0.2308	0.1387	[−0.0469, 0.4906]	0.77	1.1
	ϕ	0.9749	0.0074	[0.9588, 0.9879]	0.78	64.9
	τ^2	0.0220	0.0053	[0.0132, 0.0338]	0.78	135.7

Table 3
Estimation results of SV-RV model using realized volatilities. The last two columns are p-value of Geweke's convergence diagnostic (CD) test and inefficiency factor, respectively.

		Mean	Stdev	95% interval	CD	Inef.
SV-RV	σ^2	0.0777	0.0054	[0.0675, 0.0886]	0.80	28.2
(RV^1)	μ	−0.9179	0.0495	[−1.0157, −0.8247]	0.41	1.4
	ϕ	0.8812	0.0148	[0.8515, 0.9093]	0.71	20.7
	τ^2	0.0713	0.0070	[0.0580, 0.0858]	0.84	39.8
SV-RV	σ^2	0.1335	0.0091	[0.1154, 0.1514]	0.28	19.5
(RV^5)	μ	−0.7770	0.00561	[−0.8862, −0.6663]	0.98	1.3
	ϕ	0.8727	0.0154	[0.8413, 0.9018]	0.05	14.5
	τ^2	0.1092	0.0110	[0.0895, 0.1322]	0.02	29.2
SV-RV	σ^2	0.1930	0.0107	[0.1723, 0.2143]	0.17	12.6
(RV^{10})	μ	−0.7130	0.0577	[−0.8268, −0.6000]	0.18	1.5
	ϕ	0.8800	0.0155	[0.8481, 0.9093]	0.16	15.6
	τ^2	0.1032	0.0110	[0.0835, 0.1261]	0.11	26.0

Table 4
Estimation results of SV-RVC model using realized volatilities. The last two columns are p-value of Geweke's convergence diagnostic (CD) test and inefficiency factor, respectively.

		Mean	Stdev	95% interval	CD	Inef.
SV-RVC	ξ	-1.2344	0.0299	$[-1.2905, -1.1780]$	0.34	36.7
(RV^1)	σ^2	0.0940	0.0047	[0.0851, 0.1036]	0.24	27.8
	μ	0.2283	0.0802	[0.0666, 0.3875]	0.45	4.6
	ϕ	0.9534	0.0093	[0.9341, 0.9709]	0.05	31.0
	τ^2	0.0250	0.0038	[0.0179, 0.0327]	0.06	58.7
SV-RVC	ξ	-1.0707	0.0324	$[-1.1341, -1.0057]$	0.25	33.4
(RV^5)	σ^2	0.1467	0.0080	[0.1317, 0.1627]	0.27	20.5
	μ	0.1899	0.0785	[0.0375, 0.3478]	0.11	6.7
	ϕ	0.9294	0.0118	[0.9053, 0.9509]	0.22	20.7
	τ^2	0.0560	0.0078	[0.0418, 0.0717]	0.17	33.6
SV-RVC	ξ	-1.0437	0.0312	$[-1.1062, -0.9816]$	0.77	23.5
(RV^{10})	σ^2	0.1877	0.0105	[0.1668, 0.2081]	0.06	46.0
	μ	0.1957	0.0753	[0.0441, 0.3469]	0.55	4.7
	ϕ	0.9208	0.0139	[0.8920, 0.9460]	0.03	51.3
	τ^2	0.0631	0.0100	[0.0460, 0.0858]	0.02	76.6

Table 5
Estimation results of SV-SRV model using scaled realized volatilities. The last two columns are p-value of Geweke's convergence diagnostic (CD) test and inefficiency factor, respectively.

		Mean	Stdev	95% interval	CD	Inef.
SV-SRV	σ^2	0.0948	0.0046	[0.0854, 0.1037]	0.31	20.0
(SRV^1)	μ	0.2912	0.0741	[0.1425, 0.4337]	0.42	1.1
	ϕ	0.9544	0.0093	[0.9347, 0.9713]	0.27	20.3
	τ^2	0.0243	0.0038	[0.0181, 0.0329]	0.24	40.5
SV-SRV	σ^2	0.1480	0.0076	[0.1336, 0.1633]	0.53	16.4
(SRV^5)	μ	0.2048	0.0726	[0.0639, 0.3476]	0.81	1.9
	ϕ	0.9312	0.0113	[0.9082, 0.9519]	0.36	16.9
	τ^2	0.0542	0.0070	[0.0403, 0.0686]	0.68	27.7
SV-SRV	σ^2	0.1869	0.0108	[0.1656, 0.2076]	0.24	34.6
(SRV^{10})	μ	0.1807	0.0693	[0.0408, 0.3186]	0.18	1.6
	ϕ	0.9202	0.0139	[0.8908, 0.9447]	0.16	36.9
	τ^2	0.0636	0.0102	[0.0468, 0.0844]	0.20	56.2

Table 6
Estimation results of SV-SRVC model using scaled realized volatilities. The last two columns are p-value of Geweke's convergence diagnostic (CD) test and inefficiency factor, respectively.

		Mean	Stdev	95% interval	CD	Inef.
SV-SRVC	ξ	0.0655	0.0296	[0.0076, 0.1226]	0.90	38.0
(SRV^1)	σ^2	0.0941	0.0045	[0.0851, 0.1029]	0.68	15.3
	μ	0.2279	0.0799	[0.0682, 0.3843]	0.87	7.2
	ϕ	0.9535	0.0090	[0.9344, 0.9703]	1.00	14.9
	τ^2	0.0249	0.0036	[0.0187, 0.0327]	0.65	34.1
SV-SRVC	ξ	0.0156	0.0321	[−0.0472, 0.0793]	0.11	40.6
(SRV^5)	σ^2	0.1468	0.0080	[0.1319, 0.1628]	0.27	18.3
	μ	0.1902	0.0787	[0.0359, 0.3468]	0.03	7.8
	ϕ	0.9294	0.0118	[0.9055, 0.9510]	0.07	18.0
	τ^2	0.0560	0.0078	[0.0416, 0.0718]	0.08	29.0
SV-SRVC	ξ	−0.0214	0.0306	[−0.0811, 0.0395]	0.37	25.8
(SRV^{10})	σ^2	0.1882	0.0105	[0.1672, 0.2089]	0.14	42.3
	μ	0.1998	0.0761	[0.0504, 0.3506]	0.27	5.1
	ϕ	0.9216	0.0140	[0.8931, 0.9482]	0.02	44.9
	τ^2	0.0622	0.0101	[0.0438, 0.0850]	0.03	70.0

Table 7
Log marginal likelihoods for SV-RV and SV-RVC models with different data set. Standard errors are in parentheses.

Data	Model	Likelihood	Prior	Posterior	Marginal
$R + RV^1$	SV-RV	-5596.52	-6.21	14.31	-5617.04
		(0.59)		(0.03)	(0.59)
	SV-RVC	-4478.83	3.06	15.08	-4490.85
		(0.12)		(0.03)	(0.12)
$R + RV^5$	SV-RV	-5784.59	-10.86	13.17	-5808.62
		(0.52)		(0.02)	(0.52)
	SV-RVC	-5026.83	-4.66	13.57	-5045.07
		(0.18)		(0.02)	(0.18)
$R + RV^{10}$	SV-RV	-5953.46	-10.06	12.91	-5976.43
		(0.41)		(0.01)	(0.41)
	SV-RVC	-5264.50	-6.00	13.20	-5283.70
		(0.11)		(0.04)	(0.11)
$R + SRV^1$	SV-SRV	-4481.40	5.43	15.21	-4491.19
		(0.13)		(0.01)	(0.13)
	SV-SRVC	-4478.62	3.17	15.08	-4490.54
		(0.12)		(0.03)	(0.12)
$R + SRV^5$	SV-SRV	-5026.92	-2.18	13.73	-5042.83
		(0.16)		(0.03)	(0.16)
	SV-SRVC	-5026.70	-4.60	13.55	-5044.86
		(0.15)		(0.02)	(0.16)
$R + SRV^{10}$	SV-SRV	-5264.80	-3.98	13.37	-5283.16
		(0.14)		(0.03)	(0.14)
	SV-SRVC	-5264.44	-5.80	13.29	-5283.53
		(0.16)		(0.05)	(0.17)

Fig. 1. Realized volatility calculated from 1-minute intraday returns (RV^1, top left), scaled one (SRV^1, top right), and their logarithms ($\log RV^1$, bottom left; $\log SRV^1$, bottom right) during the period from April 1, 1996 to March 31, 2005 (2216 trading days).

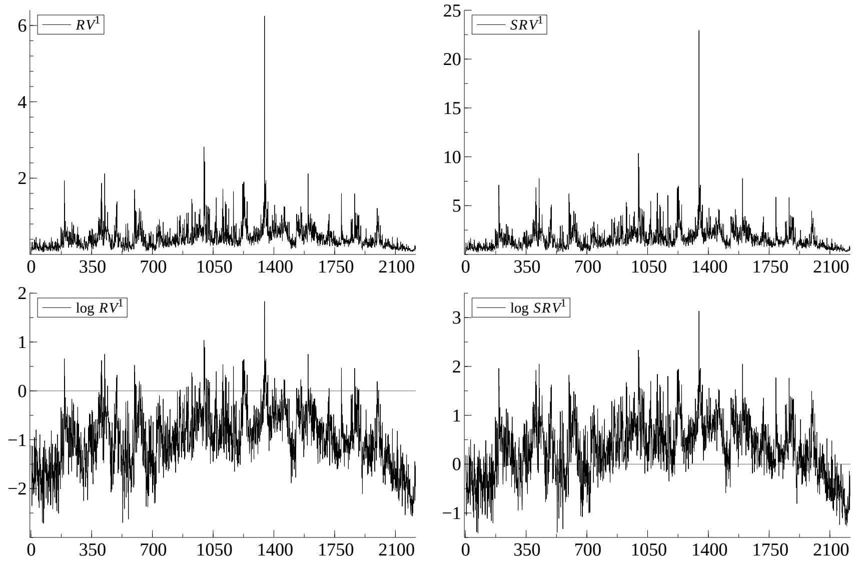

Fig. 2. Realized volatility calculated from 5-minute intraday returns (RV^5, top left), scaled one (SRV^5, top right), and their logarithms ($\log RV^5$, bottom left; $\log SRV^5$, bottom right) during the period from April 1, 1996 to March 31, 2005 (2216 trading days).

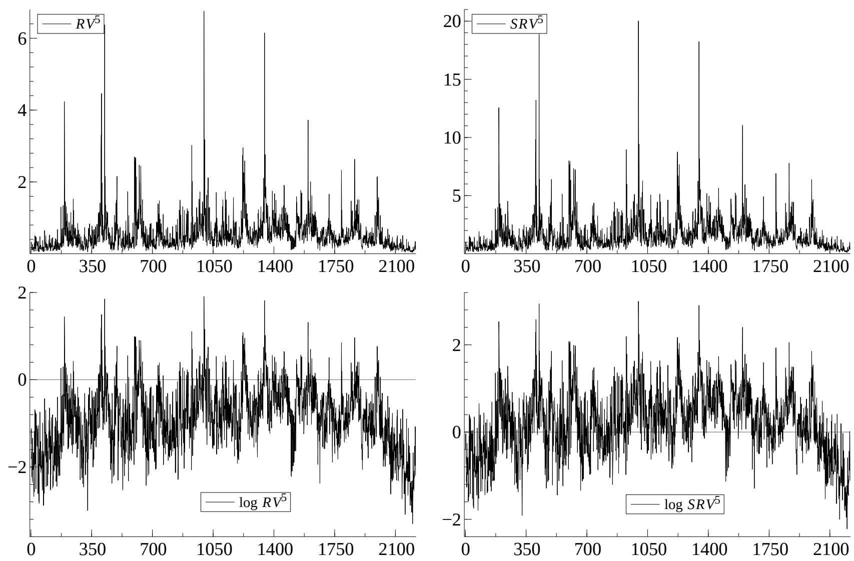

Fig. 3. Realized volatility calculated from 10-minute intraday returns (RV^{10}, top left), scaled one (SRV^{10}, top right), and their logarithms ($\log RV^{10}$, bottom left; $\log SRV^{10}$, bottom right) during the period from April 1, 1996 to March 31, 2005 (2216 trading days).

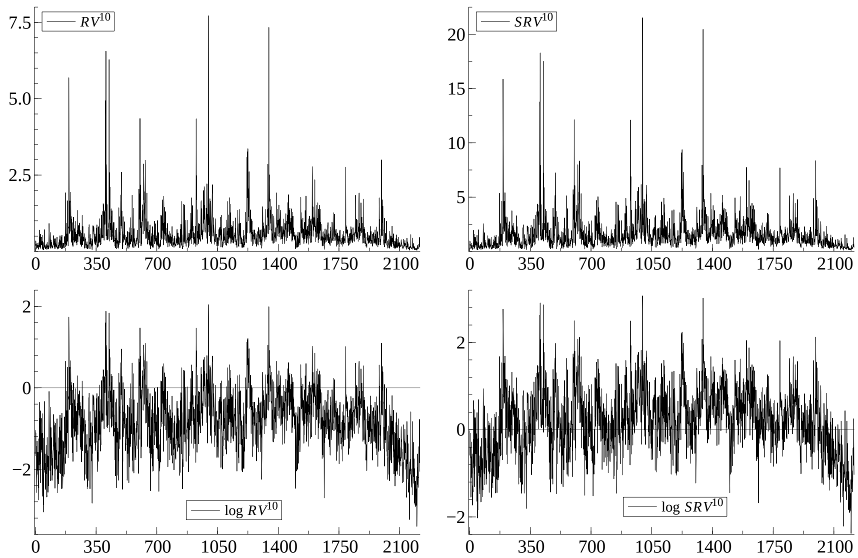

Fig. 4. daily return (R, top left), its absolute value ($|R|$, top right), squared return (R^2, bottom left), and its logarithm ($\log R^2$, bottom right) during the period from April 1, 1996 to March 31, 2005 (2216 trading days).