A Corrigendum to "Games with Imperfectly Observable Actions in Continuous Time"

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Abstract

Sannikov (2007) investigates properties of perfect public equilibria in continuous time repeated games. This note points out that the proof of the main theorem (Theorem 2) needs some corrections. I show that the main theorem holds as it is with suitable modifications of Lemmata 5 and 6.

1 Introduction

A seminal paper of Sannikov (2007) investigates properties of perfect public equilibria in repeated games with imperfect public monitoring under the continuous time setting. Theorem 2 is the main theorem of the paper, and it states that if a stochastic process of continuation payoff is on the boundary of the set of equilibrium payoffs at some time t, then it moves along the boundary after time t. A crucial step to prove the main theorem, Lemma 6, contains an error in computing a Hessian matrix. The correction of the Hessian also requires a modification of Lemma 5. The present note shows that the main theorem holds with those modifications.

2 Computation of the Hessian

Here we compute the Hessian matrix of the function f which is defined in page 1325 of Sannikov (2007). For simplicity, we consider a special case in which $\mathbf{N}$ is assumed to be $(1, 0)$. Note that although this is a special case, the obtained result is applicable to the general case by rotating $\mathcal{C}'$ properly.

Consider a twice continuously differentiable function $g : U \to \mathbb{R}$, where U is an open interval containing 0, such that $g(0) = g'(0) = 0$ and $g'' \leq 0$. Take a constant $\bar{f} > 0$, and a unit vector $\tilde{\mathbf{N}} = (n_1, n_2)$ with $n_1 \neq 0$ and $n_2 > 0$. Let $\mathcal{R}$ be the interior of the set of $x \in \mathbb{R}^2$ such that $x - (\alpha - \bar{f})\tilde{\mathbf{N}} \in \operatorname{graph} g$ with some $\alpha > 0$. In this setting, f is defined as a mapping from $\mathcal{R}$ to $[0, \infty)$ whose value is

$$f(x) = \min\{\alpha \geq 0 \mid x - (\alpha - \bar{f})\tilde{\mathbf{N}} \in \operatorname{graph} g\}. \tag{1}$$

Additionally, we define a function $h : \mathcal{R} \to \mathbb{R}$ whose value is $h(x) = x_1 - (\alpha(x) - \bar{f})n_1$, where $\alpha(x)$ is the minimal α used in the definition of $f(x)$.

First, we derive the first order derivatives of $f(\bar{x})$ and $h(\bar{x})$ for each fixed $\bar{x} \in \mathcal{R}$. In computing first order properties of f, it is justified to first order approximate g at $h(\bar{x})$, since the error induced by the approximation is at most second order[1]. For fixed $\hat{x} \in \mathcal{R}$, let $x^*(\hat{x}) = (x_1^*(x), x_2^*(x))$ be the

[1]More formally, it is because the procedure of first order approximation is independent of the choice of coordinate system, if $\operatorname{graph} g$ is locally represented as a function.

solution of the following system of equations:

$$x_2 = \hat{x}_2 + \left(\frac{n_2}{n_1}\right)(x_1 - \hat{x}_1), \tag{2}$$

$$x_2 = g(h(\bar{x})) + g'(h(\bar{x}))(x_1 - h(\bar{x})). \tag{3}$$

Then $f(x)$ is represented by $f(x) = \{x_1 - x_1^*(x)\}/n_1 + o(||x - \bar{x}||)$, where $o(||x - \bar{x}||)$ is the Landau symbol, and $h(x)$ satisfies $f(x) = \{x_1 - h(x)\}/n_1$ and thus $h(x) = x_1^*(x) + o(||x - \bar{x}||)$. Using these equations, we obtain the first order derivatives of $f(\bar{x})$ and$h(\bar{x})$ by

$$Df(\bar{x}) = \begin{bmatrix} 1/n_1 \\ 0 \end{bmatrix} - \frac{1}{n_2 - n_1 g'(h(\bar{x}))}\begin{bmatrix} n_2/n_1 \\ -1 \end{bmatrix} \tag{4}$$

$$Dh(\bar{x}) = \frac{1}{(n_2/n_1) - g'(h(\bar{x}))}\begin{bmatrix} n_2/n_1 \\ -1 \end{bmatrix}. \tag{5}$$

When $\bar{x} = 0$, $f(x)$ becomes $f(0) = (0, 1/n_2)$ by $h(0) = 0$ and $g'(0) = 0$, which is consistent with the second last equation in page 1325 since n_2 is interpreted as $\mathbf{N}\hat{\mathbf{N}}^\top = \mathbf{T}\hat{\mathbf{T}}^\top$.

Now we derive the second order derivative of f. Using (4), we obtain

$$Hf(x) = \frac{g''(h(x))}{\{n_2 - n_1 g'(h(\bar{x}))\}^2}\begin{bmatrix} -(n_2/n_1)h_1(x) & -(n_2/n_1)h_2(x) \\ h_1(x) & h_2(x) \end{bmatrix}, \tag{6}$$

where $h_i = \partial h/\partial x_i$. Since $Dh(0) = (1, -n_1/n_2)^\top$,

$$Hf(0) = \frac{-g''(0)}{n_2}\begin{bmatrix} 1 & (-n_1/n_2) \\ (-n_1/n_2) & (-n_1/n_2)^2 \end{bmatrix}. \tag{7}$$

Applying it to the general case, we get

$$Hf(W_t) = \frac{\kappa(W_t)}{\mathbf{T}\hat{\mathbf{T}}^\top}\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}, \tag{8}$$

where $\alpha = -\mathbf{T}\hat{\mathbf{N}}^\top/\mathbf{T}\hat{\mathbf{T}}^\top$.

3 Elimination of Drift

With the newly derived Hessian matrix (8), we cannot continue the rest of the proof, since Lemma 5 cannot provide sufficiently good upper bounds.

Here, we provide another inequality which can restore the argument. We denote $\mathbf{T}B_t = \phi_t$ and $\mathbf{N}B_t = \chi_t$ as in Lemma 5, and assume $\hat{\mathbf{T}}\mathbf{T}^\top > 0$ at any points of $\mathcal{C}'$, which is also implicitly assumed in Sannikov (2007).

The actual drift term of $df(W_t)$ is

$$rf(W_t) + \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top}\left\{-\mathbf{N}(g(A_t) - v) + \frac{\kappa r}{2}(|\phi_t|^2 + 2\alpha\phi_t\chi_t^\top + \alpha^2|\chi_t|^2)\right\} \tag{9}$$

$$\geq rf(W_t) + \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top}\left\{-\mathbf{N}(g(A_t) - v) + \frac{\kappa r}{2}(|\phi_t| - |\alpha||\chi_t|)^2\right\}. \tag{10}$$

When $\mathbf{N}(g(A_t)-v) \geq 0$, (10) is greater than $rf(W_t)$. When $\mathbf{N}(g(A_t)-v) < 0$, by the optimality equation of Sannikov (2007), (10) is equal to

$$rf(W_t) - \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top}\mathbf{N}(g(A_t) - v)\left\{1 - \frac{(|\phi_t| - |\alpha||\chi_t|)^2}{|\phi(a, \mathbf{T})|^2}\right\}. \tag{11}$$

In order to show that the drift term (9) is always larger than $rf(W_t)$, it is needed to eliminate the second term of (11). If a stochastic process α is constantly 0, Lemma 5 could give a good upper bound to the second term, and Girsanov's theorem could delete it using an equivalent measure. In the general case we are considering, we use the following inequality instead of Lemma 5:

$$K|\chi_t| \geq 1 - \frac{(|\phi_t| - |\alpha||\chi_t|)^2}{|\phi(A_t, \mathbf{T})|^{\mathbf{2}}}. \tag{12}$$

with some constant K. If we admit this fact, we can obtain

$$(11) \geq rf(W_t) - \frac{r}{\mathbf{T}\hat{\mathbf{T}}^\top}\{K\mathbf{N}(g(A_t) - v)|\chi_t|\} \tag{13}$$

$$\geq rf(W_t) - \frac{rL}{\mathbf{T}\hat{\mathbf{T}}^\top}|\chi_t|, \tag{14}$$

where $L = K|\mathcal{V}|$. Since the volatility of $df(W_t)$ is χ_t, we can eliminate $|\chi_t|$'s term using Girsanov's theorem; this procedure is Lemma 7.

Now, we prove the inequality (12). The following is already proved in the proof of Lemma 5:

$$J|\chi_t| \geq 1 - \frac{|\phi_t|}{|\phi(a, \mathbf{T})|} \tag{15}$$

with some constant J. By this inequality, we obtain

$$J|\chi_t| + \frac{|\alpha||\chi_t|}{|\phi(a, \mathbf{T})|} \geq 1 - \frac{|\phi_t| - |\alpha||\chi_t|}{|\phi(a, \mathbf{T})|}. \tag{16}$$

The left-hand side has an upperbound $\{J + \overline{\alpha}/\underline{\phi}\}|\chi|$, where $\overline{\alpha}$ is the largest $|\alpha| = |\mathbf{T}\hat{\mathbf{N}}^\top/\mathbf{N}\hat{\mathbf{N}}^\top|$ within $\mathcal{C}'$ and $\underline{\phi} = \inf_{a' \notin \mathcal{A}^\mathcal{N},\, \mathbf{T}'} |\phi(a', \mathbf{T}')|$. $\underline{\phi}$ is strictly positive because of Lemma 3.[2] The right-hand side has a lowerbound

$$\frac{1}{2}\left\{1 - \frac{(|\phi| - |\alpha||\chi|)^2}{|\phi(a, \mathbf{T})|^2}\right\}. \tag{17}$$

Letting $K = 2\{J + \overline{\alpha}/\underline{\phi}\}$, we obtain

$$K|\chi_t| \geq 1 - \frac{(|\phi| - |\alpha||\chi_t|)^2}{|\phi(a, \mathbf{T})|^2}. \tag{18}$$

References

[1] Sannikov, Y. (2007): "Games with Imperfectly Observable Actions in Continuous Time," *Econometrica*, 75, 1285-1329.

[2] $\max_i |\psi_i(a)|$ is strictly positive for $a \notin \mathcal{A}^\mathcal{N}$ and $|t_i| \leq 1$. Thus $|\phi(a, \mathbf{T})|$ is larger than $\min_{a \notin \mathcal{A}^\mathcal{N}} \max_i |\psi_i(a)| > 0$.