A Limit Characterization of Belief-Free Equilibrium Payoffs in Repeated Games with Almost-Perfect Monitoring*

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Abstract

The present paper provides a limit characterization of the payoff set supported by belief-free equilibria in repeated games with private monitoring, as the discount factor approaches one and the noise on private information vanishes. Contrary to the conjecture by Ely, Hörner, and Olszewski (2005), in many of the three-or-more player games, the payoff set is given by a union of product sets, as in two-player games. As an application of this result, the folk theorem is proved in N-player prisoner's dilemma games.

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1 Introduction

It is well-known that agents in a long-term relationship frequently manage to cooperate even when they are self-interested. A theory of repeated games supplies a formal framework to investigate this interesting phenomenon. Fudenberg and Maskin (1986) explain why agents are willing to cooperate by proving the folk theorem, i.e., any feasible and individually rational payoff vector can be achieved by a subgame-perfect equilibrium in an infinitely repeated game if players are patient. The folk theorem is generalized by Fudenberg, Levine, and Maskin (1994) to the *imperfect public monitoring* case in which players cannot observe the opponents' action directly, but receive noisy public information in every period.

On the other hand, the analysis becomes rather difficult under *imperfect private monitoring*, where each player receives noisy private information about the past play. In this setting, players cannot possess any common knowledge, which prevents utilizing the techniques developed in the study of public monitoring.[1] Thus, despite its potential applicability (e.g., secret price-cutting games proposed by Stigler (1964)), many problems still remain to be solved on the study of private monitoring.[2]

This paper explores the possibility of cooperation under private monitoring by examining a restricted class of sequential equilibria, which is called *belief-free*. A sequential equilibrium is belief-free if, after every history, each player's continuation strategy is optimal independently of the opponents' history.[3] Although the set of belief-free equilibria is only a subset of sequential equilibria, research on this class of equilibria is motivated by the following reasons. First, the analysis of belief-free equilibria is generally simpler than that of the others. Indeed, a player's belief about the opponents' history is irrelevant to her best reply in a belief-free equilibrium, and hence computing these beliefs is not needed in its analysis. Sec-

[1]However, once the outside cheap-talk communication is allowed, then the techniques for the public monitoring setting can be applied, since the outcome of the communication is public information. See Ben-Porath and Kahneman (1996), Compte (1998), Kandori and Matsushima (1998), Aoyagi (2002), Fudenberg and Levine (2007), and Obara (2006).

[2]See Kandori (2002) and Mailath and Samuelson (2006) for a brief survey.

[3]A sequential equilibrium that does not satisfy this property is called *belief-based*. Sekiguchi (1997) does pioneering work on this class of equilibria, and Bhaskar and Obara (2002) and Chen (2006) generalize this approach.

ond, by definition, a belief-free equilibrium is robust to additional information acquisition. Namely, even if an exogenous shock occurs and players suddenly get some information about the opponents' history, continuing to play a belief-free equilibrium is still sequentially rational. Third, because of its tractability, a belief-free equilibrium has potential applications. For example, one can construct a *review strategy* based on the idea of a belief-free equilibrium, which exhibits good performance when monitoring technology is far from accurate (Matsushima (2004), Ely, Hörner, and Olszewski (2005), and Yamamoto (2007)). Also, Takahashi (2006) demonstrates that a belief-free equilibrium is a useful notion in the context of community enforcement as well.[4]

A belief-free equilibrium has been proposed and investigated by Piccione (2002), Ely and Välimäki (2002), Ely, Hörner, and Olszewski (2005), Kandori and Obara (2006), and Yamamoto (2007). Above all, Ely, Hörner, and Olszewski (2005, hereafter EHO) may be the most notable paper. EHO characterize the payoff set supported by belief-free equilibria in all two-person games for any discount factor and for any accuracy of the monitoring technology. Particularly, in the limit as the discount factor goes to one and the noise on private information tends to vanish (in what follows, say simply "the limit"), they derive a simple and tractable formula to calculate the equilibrium payoff set; the payoff set is expressed as a union of product sets (rectangles), each of which is easily computed from the stage game payoffs.

However, EHO's analysis is limited to two-player games, and it has not been clear whether their result is generalizable to N-player games. In addition, EHO conjecture that even if such a generalization is possible, the equilibrium payoff set with more than two players is *not* expressed as a union of product sets any longer. Indeed, they state

> "It is straightforward to generalize the definition of belief-free strategies and of regimes to more than two players. However, it is no longer true that, for a fixed sequence of regimes, the payoff set has a product structure. Therefore ... determining the equilibrium payoff set becomes significantly harder."

[4]On the other hand, a criticism of belief-free equilibria is that they often fail to be purified by payoff perturbation. See Bhaskar (1998), Bhaskar and van Damme (2002), and Bhaskar, Mailath, and Morris (2006).

The main result of this paper addresses this question; the set of belief-free equilibrium payoffs is characterized in the limit for general N-player games. As opposed to EHO's conjecture, this result asserts that even if three or more players exist, the equilibrium payoff set *is* a union of (a feasible subset of) product sets, as in two-player games. The computed payoff set is equivalent to the feasible and individually rational payoff set in N-player prisoner's dilemma games, for which the folk theorem is proved. However, in most other games, the equilibrium payoff set is smaller than the feasible and individually rational payoff set.

The proof of this characterization result is divided into two steps. The first step proves necessity, i.e., any payoff vector supported by belief-free equilibria must be included in a certain payoff set. Then, in the second step, sufficiency is proved, i.e., any vector in this payoff set can be attained by a belief-free equilibrium. The proof of necessity is analogous to that for two-player games, invented by EHO. On the other hand, in proving sufficiency, a theoretical gap exists between two-player games and N-player games; when three or more players exist, to maintain a certain best reply to a player, all the opponents, i.e., more than two players, need to coordinate their play. This *coordination issue*, which does not appear in two-player games, prevents EHO's equilibrium from extending to N-player games (see section 5.2 for details). To overcome this problem, the present paper modifies a *block strategy* by Hörner and Olszewski (2006a, hereafter HO) so that the resulting strategy is a belief-free equilibrium. Although the payoff set supported by the modified block strategy is smaller than that by the original block strategy (indeed, HO prove the folk theorem under almost-perfect monitoring), it is shown that this class of belief-free equilibria can achieve any vector in the payoff set specified from necessity, as desired.

In this sense, most of the techniques employed in this paper are not innovative. Rather, the contribution of this paper is to find that the modified block strategies can achieve any belief-free equilibrium payoff and to provide a precise characterization of the set of these payoffs in the limit. As mentioned, the derived formula for computing the payoff set has the same expression as in EHO, despite the existence of the theoretical gap between two-player games and N-player games. Also, the block strategy modified by this paper is simpler than the original. This highlights that HO's equilibrium construction can be streamlined if its target payoff is in the belief-free equilibrium payoff set. Furthermore, discussion on the coordination issue illuminates the intuition behind some of the existing literature.

This paper proceeds as follows. Section 2 introduces the model and several definitions. In section 3, the limit characterization of belief-free equilibrium payoffs is provided. As an application of this result, section 4 proves the folk theorem for prisoner's dilemma games. Finally, section 5 contains further discussions.

2 Setup

2.1 The Model

The stage game is $\{I, (A_i, \Omega_i, g_i)_{i\in I}, \mu\}$; $I = \{1, 2, \cdots, N\}$ is the set of players, A_i is the finite set of player i's pure actions, Ω_i is the (possibly infinite) set of player i's private signals, $g_i : A_i \times \Omega_i \to \boldsymbol{R}$ is player i's profit function, and μ is the distribution of the signals. Let $A = \times_{i\in I} A_i$ and $\Omega = \times_{i\in I} \Omega_i$.

In the stage game, players choose an action profile $a = (a_1, \cdots, a_N) \in A$ simultaneously, and then a signal profile $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$ is realized according to the conditional distribution $\mu(\cdot|a)$. Player i's profit is determined by her own action and signal, and denoted by $g_i(a_i, \omega_i)$. Thus, given an action profile $a \in A$, i's expected stage game payoff is $\pi_i(a) = \int_\Omega g_i(a_i, \tilde{\omega}_i) d\mu(\tilde{\omega}|a)$. Let $\pi(a)$ represent a payoff vector $(\pi_i(a))_{i\in I}$. It may be note worthy that this paper does not assume the signal distribution to be full support.

Consider the infinitely repeated version of this game, where the discount factor is $\delta \in (0, 1)$. Denoting by a_i^τ and ω_i^τ the performed action and the observed signal in period τ, player i's history until period $t \geq 1$ is $h_i^t = (a_i^\tau, \omega_i^\tau)_{\tau=1}^t$. Let $h_i^0 = \emptyset$, and for each t, let H_i^t be the set of all h_i^t. A strategy for player i is a mapping $s_i : \bigcup_{t=0}^\infty H_i^t \to \triangle A_i$, where $\triangle A_i$ is the set of mixed actions of player i. For example, $s_i(h_i^t)[a_i]$ is the probability that player i takes an action a_i after a history h_i^t. Let S_i be the set of i's strategies, and let $S = \times_{i\in I} S_i$. Player i's expected average payoff from a strategy profile $s \in S$ is denoted by $w_i(s)$. That is, $w_i(s) = (1-\delta)E[\sum_{t=1}^\infty \delta^{t-1} \pi_i(a^t)|s]$. Public randomization is not allowed here.

This paper studies the case in which monitoring is almost perfect, namely, the noise on private information is sufficiently small. This assumption is formalized as follows; for any $\varepsilon \geq 0$, monitoring structure is *ε-perfect* if for each $i \in I$ and $a_i \in A_i$, there exists a partition of Ω_i into $\{\Omega_i(a)\}_{a_{-i}\in A_{-i}}$ such that

$\int_{\Omega_i(a)\times\Omega_{-i}} d\mu(\tilde{\omega}|a) \geq 1-\varepsilon$ for all $a_{-i} \in A_{-i}$. In words, given any action profile $a \in A$, the probability that player i observes an *erroneous* signal $\omega_i \notin \Omega_i(a)$ is less than ε.[5]

2.2 Belief-Free Equilibrium

A belief-free equilibrium is named after EHO, and its formal definition is as follows. Let $BR(s_{-i})$ represent the set of i's best response $s_i \in S_i$ against $s_{-i} \in S_{-i}$. For any strategy $s_i \in S_i$, player i's continuation strategy after a history $h_i^t \in H_i^t$ is denoted by $s_i|h_i^t$. To simplify the notation, write $s|h^t$ and $s_{-i}|h_{-i}^t$ instead of $(s_i|h_i^t)_{i\in I}$ and of $(s_j|h_j^t)_{j\neq i}$, respectively.

Definition 1. A strategy profile $s \in S$ is a *belief-free equilibrium* if and only if $s_i|h_i^t \in BR(s_{-i}|h_{-i}^t)$ for all $i \in I$, $h^t \in H^t$ and t.

In words, a strategy profile s is a belief-free equilibrium if, after every history, each player's continuation strategy is optimal independently of her opponents' history. It is clear that a belief-free equilibrium is a sequential equilibrium.

Following EHO, one can describe belief-free equilibria in terms of actions that prevail in a given period t. Letting $\mathscr{A}_i$ be a non-empty subset of A_i for each $i \in I$, $\mathscr{A} \equiv \times_{i\in I}\mathscr{A}_i$ is called a *regime generated from* A, and let $\mathscr{J}$ represent the set of all regimes generated from A. Let $\{\mathscr{A}(t)\}_{t=1}^{\infty}$ be an infinite sequence of regimes, that is, $\mathscr{A}(t) \in \mathscr{J}$ for all t. Then, given a sequence $\{\mathscr{A}(t)\}$, a strategy profile $s \in S$ is a *belief-free equilibrium with regime sequence* $\{\mathscr{A}(t)\}$ if s is a belief-free equilibrium, and if $\mathscr{A}(t)$ is equal to

$$\{a \in A |\ \exists h^{t-1} \in H^{t-1},\ \ s(h^{t-1})[a] > 0\}$$

for all t. Intuitively, $\mathscr{A}(t)$ is the set of action profiles taken in period t with positive probability. Notice that the definition employed here is simpler than that in EHO (and both these notions are very similar).

[5] The word "erroneous signal" may be an abuse of language. In fact, if $|\Omega_i| > |A_{-i}|$, then ε-perfection cannot reject the existence of $\omega_i \notin \Omega_i(a)$ such that for any $\tilde{a} \neq a$, the signal ω_i is more likely to occur against a than against $\tilde{a}$ (i.e., player i may believe that the opponents chose the action profile a after observing $\omega_i \notin \Omega_i(a)$). However, this does not pose any problem on the analysis of this paper.

By definition, if a strategy profile $s \in S$ is a belief-free equilibrium with regime sequence $\{\mathscr{A}(t)\}$, then for each $i \in I$ and t, choosing any action a_i from $\mathscr{A}_i(t)$ in period t is optimal for player i independently of the opponents' history. In other words, player i is indifferent between the actions in $\mathscr{A}_i(t)$ at date t regardless of the past history.

In the remainder of this paper, $\alpha_i \in \triangle\mathscr{A}_i$ is interpreted as player i's mixed action the support of which is included in the set $\mathscr{A}_i \subseteq A_i$. With an abuse of notation, let $\triangle\mathscr{A}$ be $\times_{j\in I}\triangle\mathscr{A}_j$, and $\triangle\mathscr{A}_{-i}$ be $\times_{j\neq i}\triangle\mathscr{A}_j$. A stage game payoff from mixed action profile $\alpha \in \triangle\mathscr{A}$ is easily calculated, and written as $\pi_i(\alpha)$. Similarly, a payoff by playing $a_i \in A_i$ against $\alpha_{-i} \in \triangle\mathscr{A}_{-i}$ is written as $\pi_i(a_i, \alpha_{-i})$.

3 Characterization

3.1 Result

This section presents the main result of this paper; the set of payoffs supported by belief-free equilibria is characterized in the limit where the discount factor goes to one and the noise on private signals vanishes. To state the theorem, the following notation is used. Let $p = (p(\mathscr{A}))_{\mathscr{A}\in\mathscr{J}}$ be a probability distribution over the set of all regimes, $\mathscr{J}$. Then, for any $p \in \triangle\mathscr{J}$, define $V(p)$ as

$$V(p) \equiv \text{co}\left\{ \sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\pi(a(\mathscr{A})) \;\middle|\; a(\mathscr{A}) \in \mathscr{A}, \;\; \forall \mathscr{A} \in \mathscr{J} \right\}$$

where co$B$ stands for the convex hull of B. In words, $V(p)$ is the set of feasible payoffs when players' action set is restricted to regime $\mathscr{A} \subseteq A$ with probability $p(\mathscr{A})$. In particular, letting $p^A \in \triangle\mathscr{J}$ be a unit vector that assigns one to the regime $\mathscr{A} = A$, the feasible payoff set is given by $V(p^A)$.

For all $i \in I$ and $\mathscr{A} \in \mathscr{J}$, define $\underline{v}_i^{\mathscr{A}}$ and $\overline{v}_i^{\mathscr{A}}$ to be

$$\underline{v}_i^{\mathscr{A}} \equiv \min_{\alpha_{-i}\in\triangle\mathscr{A}_{-i}} \max_{\alpha_i\in\triangle A_i} \pi_i(\alpha), \quad \text{and} \quad \overline{v}_i^{\mathscr{A}} \equiv \max_{\alpha_{-i}\in\triangle\mathscr{A}_{-i}} \min_{\alpha_i\in\triangle\mathscr{A}_i} \pi_i(\alpha).$$

Also, choose mixed action profiles $\underline{\alpha}^i(\mathscr{A}) = (\underline{\alpha}^i_j(\mathscr{A}))_{j\in I} \in \triangle\mathscr{A}$ and $\overline{\alpha}^i(\mathscr{A}) = (\overline{\alpha}^i_j(\mathscr{A}))_{j\in I} \in \triangle\mathscr{A}$ such that

$$\underline{v}_i^{\mathscr{A}} = \max_{\alpha_i\in\triangle A_i} \pi_i(\alpha_i, \underline{\alpha}^i_{-i}(\mathscr{A})), \quad \text{and} \quad \overline{v}_i^{\mathscr{A}} = \min_{\alpha_i\in\triangle\mathscr{A}_i} \pi_i(\alpha_i, \overline{\alpha}^i_{-i}(\mathscr{A})).$$

Intuitively, $\underline{v}_i^{\mathscr{A}}$ is the punishment payoff of player i when the opponents' available actions are restricted to regime $\mathscr{A}_{-i} \subseteq A_{-i}$. In fact, player i's stage game payoff is not more than $\underline{v}_i^{\mathscr{A}}$, provided that the opponents take the punishment action profile $\underline{\alpha}^i_{-i}(\mathscr{A}) \in \triangle\mathscr{A}_{-i}$. On the other hand, $\overline{v}_i^{\mathscr{A}}$ is the compensation payoff of player i when players' available actions are restricted to $\mathscr{A} \subseteq A$. In fact, player i's stage game payoff cannot fall below $\overline{v}_i^{\mathscr{A}}$, provided that player i chooses an action from $\mathscr{A}_i$ and the opponents take $\overline{\alpha}^i_{-i}(\mathscr{A}) \in \triangle\mathscr{A}_{-i}$.

For all $i \in I$, let $\underline{v}_i$ be a column vector with the components $\underline{v}_i^{\mathscr{A}}$ for all $\mathscr{A} \in \mathscr{J}$, that is, $\underline{v}_i = {}^\top(\underline{v}_i^{\mathscr{A}})_{\mathscr{A}\in\mathscr{J}}$. Similarly, let $\overline{v}_i = {}^\top(\overline{v}_i^{\mathscr{A}})_{\mathscr{A}\in\mathscr{J}}$. Notice that, for any $p \in \triangle\mathscr{J}$, $p\underline{v}_i$ is equal to $\sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\underline{v}_i^{\mathscr{A}}$, and $p\overline{v}_i$ is equal to $\sum_{\mathscr{A}\in\mathscr{J}} p(\mathscr{A})\overline{v}_i^{\mathscr{A}}$. Roughly, $p\underline{v}_i$ and $p\overline{v}_i$ are weighted averages of the punishment payoffs and the compensation payoffs, respectively.

This paper distinguishes the following three cases. Observe that only the payoff function, π_i, determines which case obtains, since both $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$ are easily computed from π_i.

1. (positive case) There exists $p \in \triangle\mathscr{J}$ such that the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$ is N-dimensional.

2. (negative case) For any $p \in \triangle\mathscr{J}$, the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$ is either empty or a singleton.

3. (abnormal case) There exists no $p \in \triangle\mathscr{J}$ such that the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$ is N-dimensional, but there exists $p \in \triangle\mathscr{J}$ such that the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]$ is neither empty nor a singleton.

Given a stage game, let V^* be the closure of the set of belief-free equilibrium payoffs in the limit where (δ,ε) converges to $(1,0)$. Formally, V^* is the closure of the set of the payoff vector $v = (v_1,\cdots,v_N)$ such that there exist $\overline{\delta}$ and $\overline{\varepsilon}$ such that, for all $\delta \in (\overline{\delta},1)$ and for all $\varepsilon \in (0,\overline{\varepsilon})$, there exists a belief-free equilibrium $s \in S$ that satisfies $w_i(s) = v_i$ for all $i \in I$. The main result of this paper is:

Theorem 1. *In the positive case,*

$$V^* = \bigcup_{\{p\in\triangle\mathscr{J}\,\mid\, p\overline{v}_i \geq p\underline{v}_i,\ \forall i\in I\}} (V(p) \cap \times_{i\in I}[p\underline{v}_i,\ p\overline{v}_i]); \tag{1}$$

In the negative case, V^* equals the convex hull of the set of (possibly mixed) Nash equilibrium payoffs in the stage game.

This theorem completely characterizes the limit of the set of belief-free equilibrium payoffs in the positive case and in the negative case. In the positive case, the payoff set is represented as a union of the terms in the parenthesis (the intersection of the feasible set with respect to p and the product set). In the negative case, there exist only trivial belief-free equilibria in which players play a static equilibrium in every period. The shape of the payoff set in the abnormal case is unknown. See section 5 for further discussions.

The theorem is established as a corollary of the following two propositions, the proofs of which are found in the next two sections.

Proposition 1. *In the positive case, V^* is included in the right-hand side of (1). In the negative case, V^* equals the convex hull of the payoff vectors of the Nash equilibria in the stage game.*

Proposition 2. *In the positive case, V^* includes the right-hand side of (1).*

3.2 Proof of Proposition 1

Consider the positive case, and let $s \in S$ be a belief-free equilibrium with regime sequence $\{\mathscr{A}(t)\}$. To simplify the notation, denote by $w_i(h^t)$ player i's continuation payoff after history h^t. Since s is belief-free, the continuation payoff $w_i(h^t)$ is independent of h_i^t, so that one can write $w_i(h_{-i}^t)$ instead of $w_i(h^t)$.

By definition,

$$w_i(h_{-i}^{t-1}) \geq (1-\delta) \sum_{a_{-i} \in A_{-i}} s_{-i}(h_{-i}^{t-1})[a_{-i}]\pi_i(a) \\ + \delta \sum_{a_{-i} \in A_{-i}} \int_{\Omega} s_{-i}(h_{-i}^{t-1})[a_{-i}] w_i(h_{-i}^t = (h_{-i}^{t-1}, (a_{-i}, \tilde{\omega_{-i}}))) d\mu(\tilde{\omega}|a) \quad (2)$$

for all $h_{-i}^{t-1} \in H_{-i}^{t-1}$ and $a_i \in A_i$ with equality if $a_i \in \mathscr{A}_i(t)$.

For each t,let $\overline{w}_i^t \equiv \max_{h_{-i}^{t-1} \in H_{-i}^{t-1}} w_i(h_{-i}^{t-1})$. In words, $\overline{w}_i^t$ is the best continuation payoff for player i from period t on. Since (2) holds with equality for all $a_i \in \mathscr{A}_i(t)$ and $w_i(h_{-i}^t)$ is equal to or less than $\overline{w}_i^{t+1}$,

$$w_i(h_{-i}^{t-1}) \leq (1-\delta) \min_{a_i \in \mathscr{A}_i(t)} \pi_i(a_i, s_{-i}(h_{-i}^{t-1})) + \delta \overline{w}_i^{t+1}$$

for all $h_{-i}^{t-1} \in H_{-i}^{t-1}$ and $a_i \in \mathscr{A}_i(t)$. Using the fact that the support of $s_j(h_j^{t-1})$ is included in $\mathscr{A}_j(t)$ for each $j \neq i$, one can get

$$w_i(h_{-i}^{t-1}) \leq (1-\delta)\overline{v}_i^{\mathscr{A}(t)} + \delta \overline{w}_i^{t+1}$$

for all $h_{-i}^{t-1} \in H_{-i}^{t-1}$, which implies that

$$\overline{w}_i^t \leq (1-\delta)\overline{v}_i^{\mathscr{A}(t)} + \delta \overline{w}_i^{t+1}. \tag{3}$$

Similarly, let $\underline{w}_i^t \equiv \min_{h_{-i}^{t-1} \in H_{-i}^{t-1}} w_i(h_{-i}^{t-1})$, which means the worst continuation payoff for player i from period t. From (2) and $w_i(h_{-i}^t) \geq \underline{w}_i^{t+1}$, one can obtain

$$w_i(h_{-i}^{t-1}) \geq (1-\delta)\max_{a_i \in A_i} \pi_i(a_i, s_{-i}(h_{-i}^{t-1})) + \delta \underline{w}_i^{t+1}$$

for all $h_{-i}^{t-1} \in H_{-i}^{t-1}$ and $a_i \in A_i$. Then, as in the above argument,

$$\underline{w}_i^t \geq (1-\delta)\underline{v}_i^{\mathscr{A}(t)} + \delta \underline{w}_i^{t+1}. \tag{4}$$

Using (3) and (4) repeatedly, one can find that i's continuation payoff $w_i(h^t)$ is in the interval

$$\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t-1}(1-\delta)\underline{v}_i^{\mathscr{A}(\tau)}, \; \sum_{\tau=t+1}^{\infty} \delta^{\tau-t-1}(1-\delta)\overline{v}_i^{\mathscr{A}(\tau)} \right]$$

for any $i \in I$, $h^t \in H^t$, and $t = \{0, 1, \cdots\}$. Equivalently, letting $p^t \in \triangle \mathscr{J}$ be the distribution satisfying

$$p^t(\mathscr{A}) = \sum_{\{\tau \mid \tau > t, \, \mathscr{A}(\tau) = \mathscr{A}\}} \delta^{\tau-t-1}(1-\delta)$$

for all $\mathscr{A} \in \mathscr{J}$, $w_i(h^t)$ is in the interval $[p^t \underline{v}_i, p^t \overline{v}_i]$. In addition, from feasibility, $(w_i(h^t))_{i \in I}$ must be an element of $V(p^t)$. Hence, $(w_i(h^t))_{i \in I}$ is in the intersection of $V(p^t)$ and $\times_{i \in I}[p^t \underline{v}_i, p^t \overline{v}_i]$, and in particular, $(w_i(s))_{i \in I}$ is in the intersection of $V(p^0)$ and $\times_{i \in I}[p^0 \underline{v}_i, p^0 \overline{v}_i]$. This confirms that the right-hand side of (1) includes V^* in the positive case.

Next, consider the negative case. Pick an arbitrary belief-free equilibrium $s \in S$. Then, as in the positive case, for any $h^t \in H^t$ and t, the continuation payoff vector $(w_i(h^t))_{i \in I}$ is in the intersection of $V(p^t)$ and $\times_{i \in I}[p^t \underline{v}_i, p^t \overline{v}_i]$, which must be a singleton in the negative case. This asserts that no dynamic incentive is supplied in the equilibrium, and hence after any history h^t, player i's action $s_i(h_i^t)$ must be a best reply to the opponents' action $s_{-i}(h_{-i}^t)$. In other words, players play a Nash equilibrium after any history. Therefore, V^* is included in the convex hull of the payoff vectors of static equilibria. The opposite inclusion is obvious.

3.3 Proof of Proposition 2

Fix an arbitrary payoff vector $v = (v_1, \cdots, v_N)$ from the interior of the right-hand side of (1). The existence of the interior is guaranteed, since this is the positive case. To prove Proposition 2, one needs to construct a belief-free equilibrium achieving v for sufficiently large δ and small ε.

Throughout the proof, let "player $i-1$" refer to player $i-1$ for each $i \in \{2, \cdots, N\}$, and to player N for $i = 1$. Define $X_i = \{G, B\}$, and let $X = \times_{i \in I} X_i$. As explained later, X_i is the message space of player i; G is called a good message, and B is a bad message.

3.3.1 Regimes, Actions and Payoffs

Pick a distribution $p \in \triangle \mathscr{J}$ such that the interior of $V(p) \cap \times_{i \in I}[p\underline{v}_i,\ p\overline{v}_i]$ involves the payoff vector v and such that $p(\mathscr{A})$ is a rational number for all $\mathscr{A} \in \mathscr{J}$.[6] Fix real numbers $\underline{w}_i$ and $\overline{w}_i$ for each $i \in I$ such that $\underline{w}_i < v_i < \overline{w}_i$ for all $i \in I$, and such that hyperrectangle $\times_{i \in I}[\underline{w}_i,\ \overline{w}_i]$ is included in the interior of $V(p) \cap \times_{i \in I}[p\underline{v}_i,\ p\overline{v}_i]$. See Figure 1.

Since each entry of p is a rational number, one can pick a corresponding sequence of regimes $(\mathscr{A}^1, \cdots, \mathscr{A}^L)$ satisfying

$$p(\mathscr{A}) = \frac{\sharp\{k \in \{1, \cdots, L\} | \mathscr{A}^k = \mathscr{A}\}}{L} \tag{5}$$

for all $\mathscr{A} \in \mathscr{J}$. By definition, for all $i \in I$,

$$\frac{1}{L}(\underline{v}_i^{\mathscr{A}^1} + \cdots + \underline{v}_i^{\mathscr{A}^L}) = p\underline{v}_i, \quad \text{and} \quad \frac{1}{L}(\overline{v}_i^{\mathscr{A}^1} + \cdots + \overline{v}_i^{\mathscr{A}^L}) = p\overline{v}_i,$$

[6] The existence of such p is proved as follows. Since this is the positive case, there exists $\hat{p} \in \triangle \mathscr{J}$ such that the intersection of $V(\hat{p})$ and $\times_{i \in I}[\hat{p}\underline{v}_i,\ \hat{p}\overline{v}_i]$ is N-dimensional. Pick an arbitrary vector $\hat{v}$ from the interior of this intersection.

For any $\lambda > 0$, pick a vector $\tilde{v}$ to satisfy $\lambda\hat{v} + (1-\lambda)\tilde{v} = v$. Observe that, as λ goes to zero, $\tilde{v}$ converges to v. Then, for sufficiently small $\lambda > 0$, $\tilde{v}$ is in the interior of the right-hand side of (1), since v is chosen from the interior. Fix such λ and $\tilde{v}$, and pick $\tilde{p} \in \triangle \mathscr{J}$ such that the intersection of $V(\tilde{p})$ and $\times_{i \in I}[\tilde{p}\underline{v}_i,\ \tilde{p}\overline{v}_i]$ includes $\tilde{v}$ (again, such $\tilde{p}$ exists, since this is the positive case). Define $p \in \triangle \mathscr{J}$ to be $p = \lambda\hat{p} + (1-\lambda)\tilde{p}$. Then, it is straightforward that v is in the interior of $V(p) \cap \times_{i \in I}[p\underline{v}_i,\ p\overline{v}_i]$, as desired. Furthermore, one can perturb this p so that each component of p is a rational number.

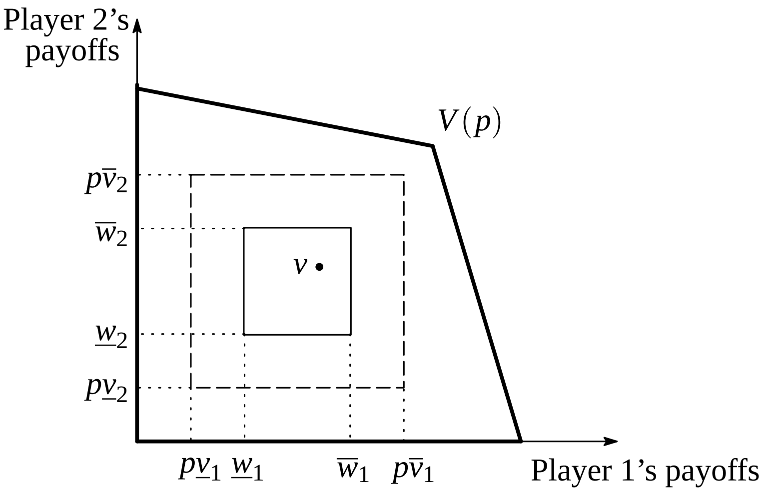

Figure 1: Payoffs

yielding

$$\frac{1}{L}(\underline{v}_i^{\mathscr{A}^1}+\cdots+\underline{v}_i^{\mathscr{A}^L})<\underline{w}_i<v_i<\overline{w}_i<\frac{1}{L}(\overline{v}_i^{\mathscr{A}^1}+\cdots+\overline{v}_i^{\mathscr{A}^L}). \tag{6}$$

For each $x\in X$, pick a sequence of action profiles $(a^{x,1},\cdots,a^{x,L})$ such that

$$a^{x,k}\in\mathscr{A}^k \tag{7}$$

for all $k\in\{1,\cdots,L\}$, and such that, letting $w^x=(w_i^x)_{i\in I}$ be the time-average payoff vector over the sequence $(a^{x,1},\cdots,a^{x,L})$,

$$w_i^x=\begin{cases}<\underline{w}_i & \text{if} \quad x_{i-1}=B\\ >\overline{w}_i & \text{if} \quad x_{i-1}=G\end{cases} \tag{8}$$

for all $i\in I$. Intuitively, (8) implies that 2^N payoff vectors $\{w^x\}_{x\in X}$ are chosen to surround the hyperrectangle $\times_{i\in I}[\underline{w}_i,\overline{w}_i]$. See Figure 2. To guarantee the existence of $(a^{x,1},\cdots,a^{x,L})$ satisfying (7) and (8), one may need to pick the parameter L large enough in determining $(\mathscr{A}^1,\cdots,\mathscr{A}^L)$.

3.3.2 Blocks and Rounds

Consider an equilibrium in which each player regards every consecutive T periods as a *block game*. The parameter T is to be determined, but basically, T is chosen sufficiently large. Every block is divided into the three kinds of rounds, a *coordination round*, a *main round*, and a *report round*.

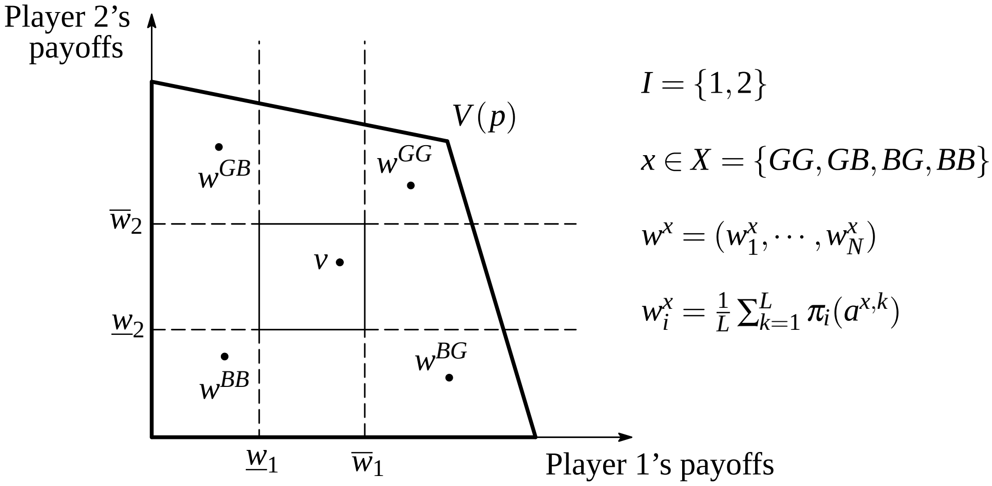

Figure 2: Actions

Coordination Round (period 1) : The actions in this round are used for communication to coordinate continuation play. The message space for player i is $X_i = \{G,B\}$. Letting A_i^G and A_i^B be a partition of A_i, player *i sends message* $x_i = G$ if she chooses an action from A_i^G; and player *i sends message* $x_i = B$ otherwise. Since monitoring is almost perfect, each player can know the message by the opponents through her private signals almost surely.

Main Round (periods $t = 2, \cdots, T-K$) : Players' behavior in this round is dependent on the communication in the coordination round. If message profile $x = (x_i)_{i \in I} \in X$ was sent in the coordination round, then players perform $(a^{x,1}, \cdots, a^{x,L})$ repeatedly during the main round. However, once player j unilaterally deviated from such behavior, they switch the actions to $\underline{\alpha}^j(\mathscr{A})$ or $\overline{\alpha}^j(\mathscr{A})$ in the remaining periods. Details are stated later.

Report Round (periods $t = T-K+1, \cdots, T$) : The report round is also used for communication, in which a player reports her private history in the coordination round and the main round. Precisely, a player reports (i) the information about what message profile was sent in the coordination round, (ii) the information about whether there exists a player who deviated unilaterally from the repetition of $(a^{x,1}, \cdots, a^{x,L})$ in the main round, and (iii) if so, the information about who is the first unilateral deviator and when it happened. The duration of the report round, K, is chosen such that all these reports can be completed by a choice of sequence of actions. As in HO, one can take K of order $\log T$. Notice that for sufficiently large T, payoffs in the communication stages (i.e., the coordination round and the report round) are negligible, since the duration of the main round is much longer than those of the coordination round and the report round.

3.3.3 Block Strategies

Let S_i^T be the set of i's strategies in the block game (T-period game), and $w_i^T(s^T)$ be i's average payoff in the block game when a strategy profile $s^T \in S^T$ is performed. That is, $w_i^T(s^T)$ is defined as $\frac{1-\delta}{1-\delta^T}E[\sum_{t=1}^T \delta^{t-1}\pi_i(a^t)|s^T]$.

Let $(\mathscr{A}^1, \cdots, \mathscr{A}^{T-K-1})$ be a sequence of regimes created by repetition of $(\mathscr{A}^1, \cdots, \mathscr{A}^L)$, that is, $\mathscr{A}^{nL+\tau} = \mathscr{A}^\tau$ for all $\tau \in \{1, \cdots, L\}$ and n. Similarly, define a sequence of actions $(a^{x,1}, \cdots, a^{x,T-K-1})$ for each $x \in X$. Let $\mathscr{S}_i^T$ be the set of i's block strategies $s_i^T \in S_i^T$ such that the support of $s_i(h_i^\tau)$ is in $\mathscr{A}_i^\tau$ for each $\tau \in \{1, \cdots, T-K-1\}$ and $h_i^\tau \in H_i^\tau$. In words, a strategy $s_i^T \in \mathscr{S}_i^T$ chooses a (possibly mixed) action from $\mathscr{A}_i^\tau$ in the τth period of the main round for all τ.

Let us define two important block strategies, good strategy $s_i^G \in S_i^T$ and bad strategy $s_i^B \in S_i^T$. In what follows, given i's history $h_i^t = (a_i^1, \omega_i^1, \cdots, a_i^t, \omega_i^t)$, let $(\hat{a}^1, \cdots, \hat{a}^t)$ represent player i's inference on the sequence of the action profiles in the past play; formally, for each $\tau \in \{1, \cdots, t\}$, profile $\hat{a}^\tau \in A$ is chosen so that $\hat{a}_i^\tau = a_i^\tau$ and $\omega_i^\tau \in \Omega_i(\hat{a}^\tau)$.[7]

In the coordination round, strategy s_i^G sends message G, and strategy s_i^B sends message B. That is,

$$s_i^G(h_i^0) \in \triangle A_i^G, \quad \text{and} \quad s_i^B(h_i^0) \in \triangle A_i^B.$$

In the report round, both strategies choose a sequence of actions to tell player i's block history, as argued in the last section. Finally, for each $h_i^t \in H_i^t$ satisfying $t \in \{1, \cdots, T-K-1\}$ (that is, periods in the main round), both strategies are defined as follows. Given $h_i^t \in H_i^t$, let

$$s_i^G(h_i^t) = s_i^B(h_i^t) = a_i^{x,t}$$

if $\hat{a}^1$ is an element of $A^x \equiv \times_{i \in I} A_i^{x_i}$ and if for any $\tau \in \{2, \cdots, t\}$, $\hat{a}^\tau$ equals $a^{x,\tau-1}$ or differs in two or more components; let

$$s_i^G(h_i^t) = s_i^B(h_i^t) = \underline{\alpha}_i^j(\mathscr{A}^t)$$

[7]Again, the word "inference" may be an abuse of language. See footnote 5.

if $\hat{a}^1 \in A^x$ with $x_{j-1} = B$, and if there exist $j \in I$ and $\tau \in \{2, \cdots, t\}$ such that (i) $\hat{a}^\tau$ differs from $a^{x,\tau-1}$ only in the jth component, and (ii) for any $\tilde{\tau} \in \{2, \cdots, \tau-1\}$, $\hat{a}^{\tilde{\tau}}$ equals $a^{x,\tilde{\tau}-1}$ or differs in two or more components; and let

$$s_i^G(h_i^t) = s_i^B(h_i^t) = \overline{\alpha}_i^j(\mathscr{A}^t)$$

if $\hat{a}^1 \in A^x$ with $x_{j-1} = G$ and if there exist $j \in I$ and $\tau \in \{2, \cdots, t\}$ satisfying (i) and (ii). Namely, given that the message profile $x \in X$ was sent in the coordination round (according to i's history), both strategies play $(a^{x,1}, \cdots, a^{x,T-K-1})$ in the main round until a unilateral deviation (again, according to i's history). Once player j unilaterally deviated in period τ, then in the remaining periods, both strategies play $(\underline{\alpha}_i^j(\mathscr{A}^\tau), \cdots, \underline{\alpha}_i^j(\mathscr{A}^{T-K-1}))$ if player $j-1$ sent B in the coordination round, and $(\overline{\alpha}_i^j(\mathscr{A}^\tau), \cdots, \overline{\alpha}_i^j(\mathscr{A}^{T-K-1}))$ if player $j-1$ sent G in the coordination round. Figure 3 depicts what will happen when players take strategy profile $s^x = (s_i^{x_i})_{i \in I}$. Note that, from (7), both s_i^G and s_i^B are elements of $\mathscr{S}_i^T$.

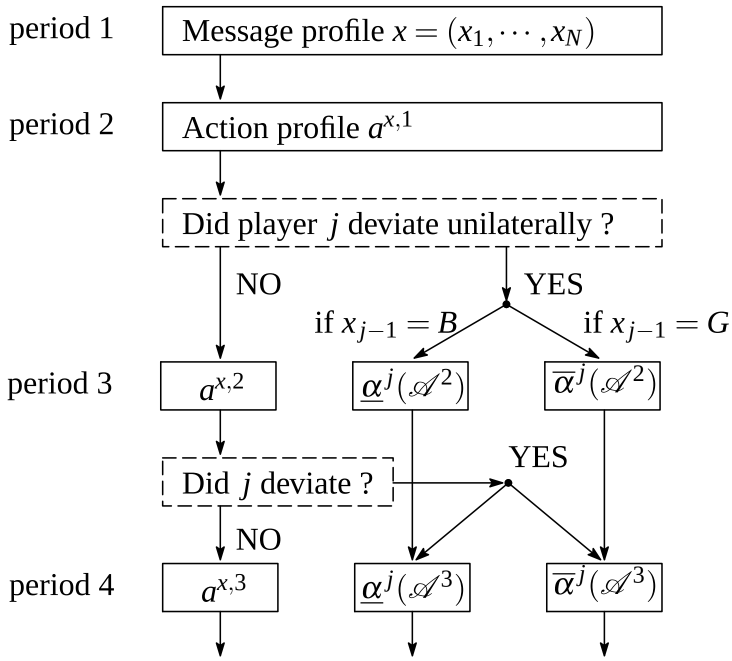

Figure 3: Strategy Profile s^x

For simplicity, assume that payoffs are not discounted and that monitoring is perfect for a while. Consider the block game in which players $-i$ play $s_{-i}^{x_{-i}} =$

$(s_j^{x_j})_{j\neq i}$ for some $x_{-i}\in X_{-i}$ satisfying $x_{i-1}=B$. In this block game, letting x_i be i's message in the coordination round, players $-i$ play $(a^{x,1},\cdots,a^{x,L})$ repeatedly in the main round unless player i deviates; and once deviated, they start to play $(\underline{\alpha}^i(\mathscr{A}^1),\cdots,\underline{\alpha}^i(\mathscr{A}^L))$ repeatedly. Then, it follows from (6) and (8) that, irrespective of i's choice of block strategy, player i earns strictly less than $\underline{w}_i$ as an average payoff in every L-period interval but at most one; the interval in which player i deviates from $(a^{x,1},\cdots,a^{x,L})$. Hence, if T is sufficiently large, then i's average payoff in the block game from s_i^T against $s_{-i}^{x_{-i}}$ cannot exceed $\underline{w}_i$ for any $s_i^T\in S_i^T$ and $x_{-i}\in X_{-i}$ satisfying $x_{i-1}=B$ (recall that payoffs in the coordination round and in the report round are negligible). Similarly, if T is sufficiently large, then i's average payoff in the block game from s_i^T against $s_{-i}^{x_{-i}}$ cannot fall below $\overline{w}_i$ for any $s_i^T\in\mathscr{S}_i^T$ and $x_{-i}\in X_{-i}$ satisfying $x_{i-1}=G$.

This argument remains valid, even if monitoring is almost perfect and the discount factor is slightly less than one. Thus, one can pick T such that there exist $\overline{\delta}\in(0,1)$ and $\overline{\varepsilon}>0$ such that for all $\delta\in(\overline{\delta},1)$ and $\varepsilon\in(0,\overline{\varepsilon})$,

$$\max_{s_i^T\in S_i^T} w_i^T(s_i^T,s_{-i}^{x_{-i}})<\underline{w}_i<\overline{w}_i<\min_{s_i^T\in\mathscr{S}_i^T} w_i^T(s_i^T,s_{-i}^{\tilde{x}_{-i}}) \tag{9}$$

for all $i\in I$, $x_{-i}\in X_{-i}$ with $x_{i-1}=B$, and $\tilde{x}_{-i}\in X_{-i}$ with $\tilde{x}_{i-1}=G$.

3.3.4 Transfers

Before moving into the analysis of infinitely repeated games, consider the following T-period game with transfers.[8] Suppose that transfer $U_i:H_{i-1}^T\to\boldsymbol{R}$ is paid to player i after period T, and call this game *auxiliary scenario*. Denote by $w_i^A(s^T,U_i)$ player i's average payoff in the auxiliary scenario when strategy profile $s^T\in S^T$ was taken and transfer U_i was paid. That is,

$$w_i^A(s^T,U_i)\equiv\frac{1-\delta}{1-\delta^T}\left\{\sum_{t=1}^{T}\delta^{t-1}E\left[\pi_i(a^t)|s^T\right]+\delta^T E\left[U_i(h_{i-1}^T)|s^T\right]\right\}.$$

For any $i\in I$, $s_{-i}^T\in S_{-i}^T$, $h_{-i}^t\in H_{-i}^t$, and U_i, let $BR^A(s_{-i}^T|h_{-i}^{t-1},U_i)$ be the set of i's best reply in the auxiliary-scenario continuation game from period $t+1$ on, given

[8]Considering such a transfer problem is helpful in analyzing the case where the discount factor goes to one. This technique is invented by Fudenberg and Levine (1994).

that the opponents play s^T_{-i} in the block game and that the opponents' past history was h^t_{-i}.

Pick transfer U^B_i as in the following lemma. Intuitively, transfer U^B_i is regarded as a subsidy to offset the difference between player i's actual block payoff and $\underline{w}_i$, the target payoff when player $i-1$ chose s^B_{i-1}.

Lemma 1. *There exist $\overline{\delta} \in (0,1)$ and $\overline{\varepsilon} > 0$ such that, for all $\delta \in (\overline{\delta}, 1)$, $\varepsilon \in (0, \overline{\varepsilon})$, and $i \in I$, there exists a transfer $U^B_i : H^T_{i-1} \to \boldsymbol{R}$ satisfying Condition 1.*

Condition 1. Transfer $U^B_i : H^T_{i-1} \to \boldsymbol{R}$ satisfies the followings.

1. For all $t \in \{0, \cdots, T-1\}$, $h^t \in H^t$, and $x \in X$ with $x_{i-1} = B$,
$$s^{x_i}_i | h^t_i \in BR^A(s^{x_{-i}}_{-i} | h^t_{-i}, U^B_i).$$
2. For all $x \in X$ with $x_{i-1} = B$, $w^A_i(s^x, U^B_i) = \underline{w}_i$.
3. For all $h^T_{-i} \in H^T_{-i}$, $0 < U^B_i(h^T_{-i}) < \frac{\overline{w}_i - \underline{w}_i}{1-\delta}$.

Proof. This lemma is proved in the same vein as Lemma 4 (a) in HO. As in HO, consider the transfer having the form

$$U^B_i(h^T_{i-1}) = \frac{1}{\delta} \left[\sum_{t=1}^{T} \delta^{t-1} \theta(h^{t-1}_{i-1}, a^t_{i-1}, \omega^t_{i-1}, I_{-i}) \right].$$

Here, $(h^{t-1}_{i-1}, a^t_{i-1}, \omega^t_{i-1})$ is a truncation of h^T_{i-1}, and I_{-i} is player $(i-1)$'s private information (contained in h^T_{i-1}) on the messages sent from players other than i during the report round (see HO for details).

First, pick nonnegative $\theta(h^{t-1}_{i-1}, a^t_{i-1}, \omega^t_{i-1}, I_{-i})$ for periods of the report round, as for periods of Phase 5 in HO. That is, θ is chosen so that player i is indifferent over all sequences of actions in the report round independently of the past history. The difficulty in applying HO's argument is that, in this paper, Ω_{i-1} may not be equal to $A_{-(i-1)}$, which violates the one-to-one correspondence between the performed action profile $a^t_{-(i-1)}$ and the observed signal ω^t_{i-1}. However, letting $\hat{a}^t_{-(i-1)}$ be the action profile satisfying $\omega^t_{i-1} \in \Omega_{i-1}(a^t_{i-1}, \hat{a}^t_{-(i-1)})$, there still exists a one-to-one correspondence between $a^t_{-(i-1)}$ and $\hat{a}^t_{-(i-1)}$. Hence, by focusing on θ depending not directly on ω_{i-1} but on $\hat{a}^t_{-(i-1)}$, one can follow HO's construction.

Similarly, define nonnegative θ for periods of the main round as for periods of Phase 4 in HO, that is, given any history up to period $t-1$, (i) player i is indifferent over all her strategies from period t on and (ii) her payoff from period t on, augmented by θ assigned from period t on, converges as $\varepsilon \to 0$ to the maximum of her payoff over all continuation strategies under perfect monitoring, conditional on the same history. Furthermore, define θ for period one, as for Phase 1 in HO.

Then, by construction, Condition 1.1 holds. Also,

$$w_i^A(s^x, U_i^B) = \max_{s_i^T \in S_i^T} w_i^T(s_i^T, s_{-i}^{x_{-i}})$$

for all $x_{-i} \in X_{-i}$ with $x_{i-1} = B$. Then, from (9), one can get U_i^B satisfying Condition 1.2 by adding to U_i^B a positive constant depending on the message profile in the coordination round (according to player $(i-1)$'s history). Finally, Condition 1.3 is obviously satisfied, provided that δ is close to one. Q.E.D.

Note that the information transmitted in the report round plays an important role in the construction of U_i^B, as argued in section 5.2.

Similarly, pick transfer U_i^G as in the following lemma. In short, transfer U_i^G is interpreted as a levy to offset the difference between player i's actual block payoff and $\overline{w}_i$, the target payoff when player $i-1$ chose s_{i-1}^G.

Lemma 2. *There exist $\overline{\delta} \in (0,1)$ and $\overline{\varepsilon} > 0$ such that, for all $\delta \in (\overline{\delta}, 1)$, $\varepsilon \in (0, \overline{\varepsilon})$, and $i \in I$, there exists a transfer $U_i^G : H_{i-1}^T \to \boldsymbol{R}$ satisfying Condition 2.*

Condition 2. Transfer $U_i^G : H_{i-1}^T \to \boldsymbol{R}$ satisfies the followings.

1. For all $t \in \{0, \cdots, T-1\}$, $h^t \in H^t$, and $x \in X$ with $x_{i-1} = G$,

$$s_i^{x_i} | h_i^t \in BR^A(s_{-i}^{x_{-i}} | h_{-i}^t, U_i^G).$$

2. For all $x \in X$ with $x_{i-1} = G$, $w_i^A(s^x, U_i^G) = \overline{w}_i$.

3. For all $h_{-i}^T \in H_{-i}^T$, $-\frac{\overline{w}_i - \underline{w}_i}{1-\delta} < U_i^G(h_{-i}^T) < 0$.

Proof. One can prove this lemma on the analogy of Lemma 1. Consider U_i^G decomposable into transfers θ. Choose nonpositive θ for periods of the report round so that player i is indifferent over all sequences of actions in the report round independently of the past history. Pick nonnegative θ for periods of the main round such that given any history up to period $t-1$, (i) player i is indifferent over all her strategies consistent with $\mathscr{S}_i^T$ (i.e., the set of $s_i^T|h_i^{t-1}$ for all $s_i^T \in \mathscr{S}_i^T$ where h_i^{t-1} denotes the history up to period $t-1$) and (ii) her payoff from period t on, augmented by θ assigned from period t on, converges as $\varepsilon \to 0$ to the minimum of her payoff over all continuation strategies consistent with $\mathscr{S}_i^T$ under perfect monitoring, conditional on the same history. In the same way, define θ for period one, and also add a negative constant depending on the message profile in the coordination round. Then, by construction, Condition 2 is satisfied. Q.E.D.

3.3.5 Equilibria

Now, consider an infinitely repeated game. The aim of this section is to show that for any $v^* \in \times_{i\in I}[\underline{w}_i, \overline{w}_i]$, there exists a belief-free equilibrium achieving v^*. This completes the proof of Proposition 2, since v is included in $\times_{i\in I}[\underline{w}_i, \overline{w}_i]$.

Fix a target payoff vector $v^* = (v_i^*)_{i\in I} \in \times_{i\in I}[\underline{w}_i, \overline{w}_i]$. Consider an equilibrium in which players regard every consecutive T periods as a block, and player i randomizes s_i^B or s_i^G in the initial period of every block. Formally, for each $i \in I$, equilibrium strategy s_{i-1}^* is implemented by the following automaton with initial state $v_i^* \in [\underline{w}_i, \overline{w}_i]$.

State w_i (for $w_i \in [\underline{w}_i, \overline{w}_i]$) : Go to phase B with probability q, and go to phase G with probability $1-q$ where q solves $w_i = q\underline{w}_i + (1-q)\overline{w}_i$.

Phase B : Play block strategy s_{i-1}^B for T periods. After that, go to state w_i given by $w_i = \underline{w}_i + (1-\delta)U_i^B(h_{i-1}^T)$ where h_{i-1}^T is her recent T-period history.

Phase G : Play block strategy s_{i-1}^G for T periods. After that, go to state w_i given by $w_i = \overline{w}_i + (1-\delta)U_i^G(h_{i-1}^T)$.

Since U_i^B and U_i^G are functions of h_{i-1}^T, player $(i-1)$'s state, w_i, is determined only by her own private history. Also, Conditions 1.3 and 2.3 assert that both $\underline{w}_i + (1-\delta)U_i^B(h_{i-1}^T)$ and $\overline{w}_i + (1-\delta)U_i^G(h_{i-1}^T)$ are in the interval $[\underline{w}_i, \overline{w}_i]$. Hence, the above automaton is well-defined. Similarly to HO's block strategy equilibrium, one-shot deviation property and Conditions 1.1, 1.2, 2.1, and 2.2 establish that strategy profile s^* is a sequential equilibrium achieving v^*. In particular, from Conditions 1.1 and 2.1, s^* is belief-free, as desired.

4 The Folk Theorem in Prisoner's Dilemma

Ely and Välimäki (2002) and Yamamoto (2007) construct a belief-free equilibrium in N-player prisoner's dilemma games, and show that a subset of the feasible and individually rational payoff set (including payoffs by mutual cooperation) is achievable. This section strengthens their results by applying Theorem 1; the folk theorem is proved in N-person prisoner's dilemma games.[9]

The stage game is a N-player prisoner's dilemma if (i) $|I| = N$, (ii) $A_i = \{C_i, D_i\}$ for all $i \in I$, (iii) $\pi_i(D_i, a_{-i}) \geq \pi_i(C_i, a_{-i})$ for all $i \in I$ and $a_{-i} \in A_{-i}$, (iv) $\pi_i(C_j, a_{-j}) \geq \pi_i(D_j, a_{-j})$ for all $i \in I$, $j \neq i$ and $a_{-j} \in A_{-j}$, and (v) $\pi_i(C) > \pi_i(D)$ for all $i \in I$ where $C = (C_1, \cdots, C_N)$ and $D = (D_1, \cdots, D_N)$. In words, the first and second conditions state that there are N players and every player chooses cooperation, C_i, or defection, D_i. The third and fourth conditions assert that defection weakly dominates cooperation and that cooperation increases the opponents' profits. Finally, the fifth condition says that the payoff from mutual cooperation exceeds the one from mutual defection.

Proposition 3. *Suppose that the stage game is a N-player prisoner's dilemma and the feasible payoff set is full dimensional, i.e., $dim V(p^A) = N$. Then, V^* is exactly equal to the set of feasible and individually rational payoff vectors.*

Proof. First, notice that a N-player prisoner's dilemma corresponds to the positive case, since $\bar{v}_i^A = \pi_i(C)$ is strictly greater than $\underline{v}_i^A = \pi_i(D)$ and the feasible payoff set is full dimensional. Fix an arbitrary vector $v = (v_1, \cdots, v_N)$ from the feasible and individually rational payoff set. It suffices to show that there exists $p \in \triangle \mathscr{J}$ such that $p\bar{v}_i \geq p\underline{v}_i$ for all i and $v \in V(p) \cap \times_{i \in I}[p\underline{v}_i, p\bar{v}_i]$.

Given any $a \in A$, let $\mathscr{A}(a)$ represent the regime $\mathscr{A} = \times_{i \in I} \mathscr{A}_i$ such that $\mathscr{A}_i = \{C_i, D_i\}$ for all $i \in I$ satisfying $a_i = C_i$ and $\mathscr{A}_i = \{D_i\}$ for all $i \in I$ satisfying $a_i = D_i$. Then, letting $\kappa = (\kappa(a))_{a \in A}$ be a probability distribution over A such that $v_i = \sum_{a \in A} \kappa(a) \pi_i(a)$ for all $i \in I$, pick $p \in \triangle \mathscr{J}$ such that $p(\mathscr{A}(a)) = \kappa(a)$ for all $a \in A$ and $p(\mathscr{A}) = 0$ for other $\mathscr{A}$. The existence of such κ is guaranteed, since v is chosen from the feasible set. By construction, $\bar{v}_i^{\mathscr{A}(a)} = \pi_i(a)$ and $\underline{v}_i^{\mathscr{A}(a)} = \pi_i(D)$ for all $i \in I$ and $a \in A$, and hence $p\bar{v}_i = v_i$ and $p\underline{v}_i = \pi_i(D)$ for all $i \in I$. This

[9] The definition of N-player prisoner's dilemma in this paper is slightly different from that in Yamamoto (2007).

proves that $p\overline{v}_i \geq p\underline{v}_i$ for all $i \in I$, since from individual rationality, v_i is not less than $\pi_i(D)$. In addition, v is obviously an element of $V(p)$. Hence, v is in the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$. Q.E.D.

The proposition asserts that the folk theorem is proved in N-player prisoner's dilemma games even when attention is restricted to belief-free equilibria. However, in general N-player games, the payoff set calculated by formula (1) is smaller than the feasible and individually rational payoff set. This is analogous to the result in two-player games; EHO show that the folk theorem holds for belief-free equilibria in two-person prisoner's dilemma games, but not in more general games.

5 Discussion

5.1 Performance of Block Strategies

In the block strategy equilibrium constructed in section 3.3, player i's continuation payoff from the beginning of every block is dependent only on player $(i-1)$'s continuation strategy. Namely, player i's continuation payoff is equal to player $(i-1)$'s state, $w_i \in [\underline{w}_i, \overline{w}_i]$, and is not dependent on other players' states. This condition is demanding compared to general belief-free equilibria, the definition of which allows each player's continuation payoff to depend on *all* the opponents' continuation strategy. Hence, one may conjecture that the payoff set sustained by the block strategy in this paper is smaller than the payoff set supported by belief-free equilibria.

Nevertheless, as a consequence of Propositions 1 and 2, these two payoff sets are identical in the limit where the discount factor goes to one and the noise vanishes. In other words, the block strategy can achieve any payoff vector supported by belief-free equilibria.

The key insight is that each player's intention (state) at the beginning of the block is revealed in the first period of the block (i.e., in the coordination round), and this information is shared by all the players in the remaining periods within the block. For example, when player $i-1$ intends to punish player i and sends the bad message in the coordination round, all the players receive this message almost perfectly, and they indeed choose the actions that yield a low payoff to player i in the subsequent periods. Namely, if player $i-1$ wants to punish player i, then i is punished by *all* the opponents throughout the block. Similarly, if player $i-1$ wants to reward player i, then i is indeed rewarded by *all* the opponents throughout the block. As a result, player $i-1$ can manipulate player i's dynamic incentive to the same extent as a coalition by all i's opponents can do, and hence the above equilibrium construction does not lose any generality in terms of dynamic incentive schemes. Therefore, the payoff vector sustained by belief-free equilibria is also supported by the block strategy equilibria.

The above argument informally explains why the analysis of this paper is limited to the almost-perfect monitoring case. To see this, suppose that monitoring is far from accurate. Then, the communication in the coordination round becomes inefficient, i.e., players receive false messages with high probability. This implies that, even when player $i-1$ intends to punish player i and sends a bad message, players fail to cooperate to punish player i with high probability. Therefore, player $i-1$ cannot manipulate player i's dynamic incentive to a satisfactory extent. Thus, it is natural to conjecture that the payoff set of the block strategy equilibria is strictly smaller than that of all belief-free equilibria. Hence, to characterize a belief-free equilibrium payoff set in these cases, one may need to find a new equilibrium construction.

5.2 Comparison with EHO

EHO characterize the payoff set sustained by belief-free equilibria in all two-player games with general information structure. In particular, when the technology of monitoring approximates perfection and the discount factor tends to one, they derive the formula

$$V^* = \bigcup_{\{p\in\triangle\mathscr{J} \mid p\overline{v}_i \geq p\underline{v}_i,\ \forall i\in I\}} \times_{i\in I}[p\underline{v}_i, p\overline{v}_i] \tag{10}$$

in the positive case. In words, the set of belief-free payoffs, V^*, is equal to the union of the product set $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$ in two-player games.

At first glance, EHO's formula seems slightly different from that of this paper; the term $V(p)$ appears in the right-hand side of (1), but not in (10). However, this difference is not essential in the following sense. Observe that, if only two players exist, then $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i] \subseteq V(p)$ for any $p \in \triangle\mathscr{J}$ satisfying $p\overline{v}_i \geq p\underline{v}_i$ for all $i \in I$.[10] This inclusion implies that the rectangle $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$ equals the intersection of $V(p)$ and $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$. Therefore, the right-hand side of (10) is exactly equal to that of (1) in two-player games. In other words, the formula (1) subsumes (10).[11]

On the other hand, EHO's equilibrium construction to establish the above characterization result is completely different from that in this paper. Namely, EHO's equilibrium strategy is implemented by a two-state automaton such that a player chooses an action punishing the opponent at one state while an action rewarding the opponent at another state, and she transits over these two states in every period to assure that the opponent has the prescribed best reply. Obviously, their equilibrium strategy is much simpler than the T-period block strategy in the present paper. This striking difference comes from the *coordination issue* unique to three-or-more player games.

To see this, consider a belief-free equilibrium with three (or more) players. By definition, in this equilibrium, players 2 and 3 coordinate their continuation play from period t on *after any pair of histories* (h_2^{t-1}, h_3^{t-1}) so that player 1 has the identical best reply. In other words, players 2 and 3 need to coordinate *with no information about one another's past history*, and hence *with no information about one another's continuation strategy*. To get rid of this problem, the present paper considers the block strategy such that a player announces her block history

[10]For example, one can confirm that $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$ is included in $V(p)$ for the unit vector p^A in general two-player games as follows. For simplicity, for each $i \in I$ and $j \neq i$, denote $\underline{\alpha}_i^j(A)$ and $\overline{\alpha}_i^j(A)$ by $\underline{\alpha}_i$ and $\overline{\alpha}_i$ respectively. By definition, $\pi_i(a_i, \underline{\alpha}_j) \leq \underline{v}_i^A$ and $\pi_i(a_i, \overline{\alpha}_j) \geq \overline{v}_i^A$ for all $a_i \in A_i$. Therefore, the convex hull of $\{(\pi_1(\alpha), \pi_2(\alpha)) | \alpha \in \{(\underline{\alpha}_1, \underline{\alpha}_2), (\underline{\alpha}_1, \overline{\alpha}_2), (\overline{\alpha}_1, \underline{\alpha}_2), (\overline{\alpha}_1, \overline{\alpha}_2)\}\}$ includes $\times_{i\in I}[\underline{v}_i^A, \overline{v}_i^A]$, which leads to the desired result.

[11]Conversely, if more than two players exist, the right-hand side of (1) often does not coincide with that of (10), since $\times_{i\in I}[p\underline{v}_i, p\overline{v}_i]$ may not be included in $V(p)$. For example, consider the following three-player game:

1,0,0	0,0,1
1,0,0	0,0,0

0,0,0	0,0,1
0,1,0	0,1,0

Here, player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. By definition, $\underline{v}_i^A = 0$ and $\overline{v}_i^A = 1$ for all $i \in I$, and hence $p^A\underline{v}_i = 0$ and $p^A\overline{v}_i = 1$ for the unit vector p^A. Obviously, $V(p^A)$ does not include $\times_{i\in I}[p^A\underline{v}_i, p^A\overline{v}_i] = [0,1]^3$.

in the report round, i.e., at the end of each block (note that this idea is borrowed from HO, as argued in the next section). In this setting, even if players 2 and 3 do not know one another's history up to the middle of the block (say, period $t-1$) and they cannot immediately coordinate the continuation play from period t, they eventually obtain precise information about the opponents' history up to period $t-1$ through the communication at the end of the block. Then, they can start coordination from the next block so that player 1 has the prescribed best reply at period t, as required.[12] Such coordination is indeed possible since, as argued in section 5.1, the coordination round and the main round are designed so that player 1 can be punished or rewarded, contingent on states of the opponents. On the other hand, EHO's equilibrium construction does not supply such a communication scheme, causing a difficulty in extending it to three-or-more player games.[13]

Exceptions are Ely and Välimäki (2002) and Yamamoto (2007), who study N-player prisoner's dilemma games and establish a belief-free equilibrium à la EHO. Precisely, their equilibrium is implemented by two-state automata such that player i chooses action C_i at one state while D_i at another state, and she transits over these two states in every period. In this construction, even if players do not know one another's history up to period $t-1$, they can immediately acquire information about one another's current state through the signal observed in period t. This enables players to coordinate the transition between periods t and $t+1$ to maintain the prescribed best reply at period t. Intuitively, the action taken today plays two roles simultaneously here: yielding a certain payoff to punish or reward the opponents (as in the coordination round and the main round in this paper), and revealing the current state (as in the report round).

[12] Mathematically, communication in the report round assures that the number of the unknowns, denoted by $\theta(h_{i-1}^{t-1}, a_{i-1}^t, \omega_{i-1}^t, I_{-i})$ in the proof of Lemmas 1 and 2, is equal to or greater than the number of the equations imposed by a player's incentive compatibility. Indeed, one can confirm that the number of θ is often less than the number of equations, if the report round is removed and θ cannot depend on the information in the report round, I_{-i}.

[13] Technically, the coordination issue explained here prevents a straightforward generalization of a notion of *strongly self-generation*, which plays a central role in the analysis of EHO. It is unknown whether or not one can modify the definition of strongly self-generation for N-player games. Also, some of the propositions derived by EHO for two player games cannot be extended to three-or-more player games, due to the coordination issue. For example, EHO show that given a regime sequence, any pair of belief-free equilibria is exchangeable in two player games. However, this result is not preserved in three-or-more player games. The proof is omitted.

Yet their method is specific to prisoner's dilemma type games, since in general N-player games, achieving a certain payoff for punishment is often incompatible with supplying a communication scheme. To see this, notice that a (possibly mixed) action minimaxing player i differs from that minimaxing player j in general. Then, to punish each of the opponents by these minimax actions (this punishment mechanism is needed if the target payoff vector is close to the minimax payoff), a player must have a state space with at least $N-1$ components, which is often larger than her action space. This undermines the communication scheme preserved in Ely and Välimäki (2002) and Yamamoto (2007), since the action taken today cannot provide enough information to identify the current state; mathematically, the number of the unknowns is often more than the number of the equations to be satisfied. To get rid of this incompatibility, one may need to introduce a sort of block strategies, i.e., strategies in which periods for communication are isolated from those for yielding a certain payoff.

5.3 Comparison with HO

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The block strategy equilibrium constructed in this paper is a modification of that by HO, who proved the folk theorem with almost-perfect monitoring. In the block strategy by HO, each player performs either a good strategy or a bad strategy in every T-period block. These strategies are chosen such that player i's best reply is independent of the past history when player $i-1$ chose the bad strategy (say, state B), but not when player $i-1$ chose the good strategy (say, state G). HO show that any feasible and individually rational payoff is supported by this class of equilibria.

The present paper modifies HO's equilibrium construction so that a player's best reply is independent of the past history not only in state B but also in state G. This modification can be accomplished by designing strategies for both states B and G as HO does for state B. By construction, the resulting strategy profile is a belief-free equilibrium, and more importantly, it can support any vector in the belief-free equilibrium payoff set, as explained in section 5.1 (although this set is smaller than that supported by HO's equilibrium).

Thus, as argued in the introduction, most of the techniques employed in this paper are not new. However, the result of this paper highlights that HO's equilibrium construction can be simplified if its target payoff is in the belief-free equilibrium payoff set. In fact, HO embeds the following tricks in their equilibrium construction; two communication stages (what they call Phases 2 and 3) are incorporated after the initial communication; and a player chooses each action with at least probability $\sqrt{\varepsilon}$ after most of the histories (what they call *regular histories*). These tricks guarantee that a player is sure about the opponents' history after every regular history. Then, her best reply in such a history is easily computed, despite the fact that a belief is relevant to a best reply in HO's equilibrium. In addition, HO needs to introduce another trick, Kakutani's fixed point theorem, to determine the play after the other histories, since a player is not sure about the opponents' history and computing her best reply is significantly hard in such histories. On the other hand, if the target payoff is in the belief-free equilibrium payoff set, one can modify a block strategy so that a player's belief is irrelevant to her best reply, as in this paper. Then, a player's belief is negligible in the equilibrium construction, and hence all the above tricks can be omitted. That is, players spend only one period for the communication at the beginning of each block, block strategies are not perturbed with $\sqrt{\varepsilon}$, and the equilibrium strategy is explicitly determined without a fixed point theorem.

Moreover, it is clarified that the block strategy is applicable to a broader class of monitoring structures than HO claim, if the target payoff is in the belief-free equilibrium payoff set. HO argue in Remark 4 that their equilibrium fails to work if more than two players exist, private signals are correlated among players to some extent, and the signal space Ω is rich in the sense that the cardinality of Ω_i is strictly greater than that of A_{-i}.[14] Loosely speaking, a player cannot maintain an appropriate belief under such a circumstance, which undermines their equilibrium construction for state G. In contrast, the modified block strategy still works under this situation, since a belief is irrelevant. Indeed, this paper requires only ε-perfection for the monitoring technology.

Finally, this paper can be regarded as a bridge connecting two papers, HO and Yamamoto (2006). When monitoring technology is not almost-perfect but

[14] An extreme example is the almost-perfect almost-public monitoring case studied in section 6 of Mailath and Morris (2002). See also Mailath and Morris (2006) and Hörner and Olszewski (2006b) for recent research on almost-public monitoring cases.

conditionally independent, one way to construct a sequential equilibrium is to combine the idea of review strategies with the existing belief-free equilibria under almost-perfect monitoring, as in Matsushima (2004), EHO, and Yamamoto (2007). Unfortunately, as argued in Remark 5 of HO, one cannot directly follow this approach to extend HO's folk theorem to the conditionally independent monitoring case, since their equilibrium is not belief-free. On the other hand, it is seemingly possible to generalize the result of this paper to the independent monitoring case, since the modified block strategy is belief-free. A subsequent work by Yamamoto (2006) confirms that this conjecture is true, and demonstrates that a payoff set similar to (1) is supported under conditionally independent monitoring. In particular, the folk theorem is obtained for N-player prisoner's dilemma game with any degree of accuracy of monitoring.

5.4 Abnormal Case

Theorem 1 is not applicable to the abnormal case. Indeed, the payoff set supported by belief-free equilibria may not be expressed by formula (1) in this case. For example, consider the following three-player game by Fudenberg and Maskin (1986):

1,1,1	0,0,0
0,0,0	0,0,0

0,0,0	0,0,0
0,0,0	1,1,1

Here, player 1 chooses a row, player 2 chooses a column, and player 3 chooses a matrix. It is easy to see that this game is included in the abnormal case.

Fudenberg and Maskin (1986) and Fudenberg, Levine, and Takahashi (2007) analyze this game under perfect monitoring and show that each player's payoff cannot fall below $\frac{1}{4}$ in any subgame-perfect equilibrium provided that the discount factor is less than one. Here a similar result will be shown under the private monitoring setting. As in the proof of Proposition 1, letting $s \in S$ be a belief-free equilibrium with regime sequence $\{\mathscr{A}(t)\}$ and $\underline{w}_i^t$ be the minimum of $w_i(h^{t-1})$,

$$w_i(h^{t-1}) \geq (1-\delta)\pi_i(a_i, s_{-i}(h_{-i}^{t-1})) + \delta \underline{w}_i^{t+1}$$

for all $i \in I$, $h^{t-1} \in H^{t-1}$, and $a_i \in A_i$. Notice that $w_i(h^{t-1})$ and $\underline{w}_i^{t+1}$ do not depend on i because $\pi_1(a) = \pi_2(a) = \pi_3(a)$ for all $a \in A$. Hence,

$$w_i(h^{t-1}) \geq (1-\delta) \max_{j \in I} \max_{a_j \in A_j} \pi_j(a_j, s_{-j}(h_{-j}^{t-1})) + \delta \underline{w}_i^{t+1}$$

for all $h^{t-1} \in H^{t-1}$. Since $\max_{i\in I} \max_{a_i \in A_i} \pi_i(a_i, \alpha_{-i}) \geq \frac{1}{4}$ for any $\alpha_{-i} \in \triangle A_{-i}$, one can get $\underline{w}_i^t \geq (1-\delta)\frac{1}{4} + \delta \underline{w}_i^{t+1}$ for all t and $i \in I$, which leads to $w_i(s) \geq \frac{1}{4}$. Therefore, the payoff set $\{(v_1, v_2, v_3) | v_i < \frac{1}{4},\ \text{for some } i \in I\}$ is not supported by belief-free equilibria. On the other hand, by definition, $\underline{v}_i^A = 0$ and $\overline{v}_i^A = \frac{1}{4}$ for all $i \in I$ so that the right-hand side of (1) includes $\{(v_1, v_2, v_3) | v_1 = v_2 = v_3,\ 0 \leq v_i \leq \frac{1}{4}\}$. This implies that the payoff set is not expressed by (1) in this example.

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