Social Preference under Uncertainty: Equality of Opportunity vs. Equality of Outcome

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Abstract

An axiomatic model of social preferences under uncertainty is developed. It is shown that this model expresses the conflict between equality of opportunity and equality of outcome and includes theory of inequality aversion (Fehr-Schmidt,1999), as a special case. Some agents choose a lottery that is equal in opportunity, even if they can draw an unequal outcome. Other agents choose a lottery that is equal in outcome, even if it is not equal in opportunity. The paper axiomatizes the simple form of utility function which includes both choices. It is shown that this model can explain consistently several different experimental results, efficiency, procedural fairness, altruism, spite as well as inequality aversion. It is also shown that the model of this paper can be alternated into a generalization of multi-period consumption model by Gilboa (1989).

KEYWORDS: Equality of Opportunity, Equality of Outcome, Independence Axiom, Preference over lotteries, Social Preference.

1 Introduction

There is overwhelming evidence that a person's welfare is affected by relative payoff (See Fehr and Schmidt (2005) for a survey). This is commonly referred to as social preferences. Many papers propose different forms of utility functions representing social preferences such as Fehr and Schmidt (1999), Bolton and Ockenfels (2000), and so on. However, when all of the authors apply their model under uncertainty, they assume expected utility theory. For example, consider a lottery

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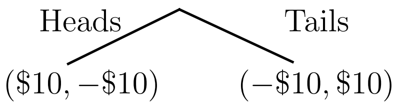

Figure 1:

$\big((\$10, -\$10), .5; (-\$10, \$10), .5\big)$ arising from a fair coin toss in which heads result in the transfer of $10 to the decision maker from the other person and tails results in the transfer of $10 from the decision maker to the other person. All of the authors assume the utility of the decision maker is

$$.5U(\$10, -\$10) + .5U(-\$10, \$10), \tag{1}$$

where U is a deterministic utility function representing social preference. U summarizes moral judgment on a pair of prizes as well as personal satisfaction from consuming his own prize. Although the forms of U vary considerably, depending on the author, what all of them assume is the *independence axiom.* For instance, Fehr and Schmidt (1999) assume expected utility theory when they analyze behavior of a principal with social preference under the uncertainty of the agents' type, or whether agents consider fairness or not. Bolton and Ockenfels (2000) also assume the responders' preference to have expected utility form in an ultimatum game under the uncertainty of pie size.

Conceptually, what makes a sharp distinction in social preferences between uncertain and deterministic situations is the fact that, under uncertainty, there are two distinct and sometimes incompatible notions of equalities, as opposed to a deterministic situation. One notion is ex ante (expected) equality and the other is ex post equality. In other words, under uncertainty, people would care about equality of opportunity as well as that of outcome. The above lottery $\big((\$10, -\$10), .5; (-\$10, \$10), .5\big)$ would be unobjectionable from the view point of equality in opportunity since the expected prize is the same for both people. However it would be objectionable from the view point of equality in outcome since each person's prize is not the same in every states. Assuming independence implies that decision maker ignores equality of opportunity. That is because, the expected utility form, such as 1, does not have a term depending on expected payoffs. In other words, expected utility form implies that the decision maker only cares about equality of outcome.

The purpose of this paper is to axiomatize the simple tractable form of a utility function representing preferences for equality of opportunity and equality of outcome. It can be proved that the form becomes a weighted average of utilities from equality of opportunity and utilities from equality of outcome. Under the assumption of risk neutrality for simplicity, the utility from the lottery $\big((x_1, x_2), p; (y_1, y_2), 1-p\big)$

is represented by[1]

$$\delta U\Big(px_1+(1-p)x_2, py_1+(1-p)y_2\Big)+(1-\delta)\Big[pU(x_1,x_2)+(1-p)U(y_1,y_2)\Big],$$

where

$$U(z_1,z_2)=z_1-\alpha\max\{z_2-z_1,0\}-\beta\max\{z_1-z_2,0\}.$$

If the parameters α and β are positive, these terms represent disutility from inequality when the decision maker is worse off and better off, respectively. If the parameters are negative, they represent the corresponding utility. The parameter δ in $[0,1]$ shows relative importance of inequality of outcome. As we will see, it can be proved that all of the parameters are unique. Moreover, constructions of parameters show that we can pin down parameters of a decision maker by employing a simple experiment. For degenerate lotteries, under the assumption of risk neutrality the representation reduces into Fehr and Schmidt (1999) utility function. This paper will show that two key axioms, *position-preserving independence* and *equivalence*, in addition to the weak ordering and the continuity axioms, are necessary and sufficient for the representation. As applications of the model, I will show that this model can explain consistently several seemingly different experimental results; procedural fairness, efficiency, altruism, as well as inequality aversion.

Position-preserving independence is a weak version of the independence axiom. The following thought experiment casts doubt on the assumption that social preference satisfies the independence axiom. Consider two pairs of prizes $(\$10,\$10)$ and $(\$10,-\$10)$. If the decision maker choose the first pair, both get $\$10$. If the decision maker chooses the second pair, the decision maker gets $\$10$ while $\$10$ is taken away from the other person. If there is no reason to punish the other person, the decision maker may prefer $(\$10,\$10)$ to $(\$10,-\$10)$. Next, consider lotteries $\big((\$10,\$10),.5;(-\$10,\$10),.5\big)$ and $\big(\$10,-\$10),.5;(-\$10,\$10),.5\big)$. The first lottery can be thought of as the lottery arising from a coin toss in which $(\$10,\$10)$ is realized if heads comes up and $(\$10,-\$10)$ is realized if tails does. Similarly, the second one would be the coin toss where heads results in $(\$10,-\$10)$ and tails results in $(\$10,-\$10)$. Which lottery does the decision maker prefer? Conditional on heads, the decision maker may choose the first one. However, in the first lottery, the decision maker's expected prize is smaller than that of the other person. Hence the decision maker may envy the other person. In the second lottery, chances that

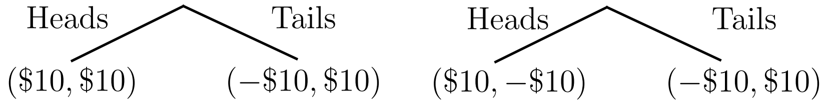

Figure 2:

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[1] $\big((x_1,x_2),p;(y_1,y_2),1-p\big)$ is a lottery which yields (x_1,x_2) with probability p and (y_1,y_2) with probability $1-p$, where the first coordinate is the prize of the decision maker and the second one is that of the other person.

the decision maker gets more than the other person are fifty-fifty and both persons' expected prizes are the same[2]. Therefore, the decision maker may not prefer $\big((\$10,\$10),.5;(-\$10,\$10),.5\big)$ to $\big((\$10,-\$10),.5;(-\$10,\$10).5\big)$ thus violating the independence axiom.

This evidence shows that it is necessary to weaken the independence axiom when we apply a theory of social preferences under uncertainty. The violation of the independence axiom is caused by the fact that, in the latter lottery, differences in payoff have opposite signs depending on which side comes up. Because of this, the difference in expected payoff is smaller, so that the decision maker prefers the latter lottery. Therefore, in order to avoid this kind of violation, two lotteries which are mixed with each other must have non-opposite sign of difference between decision maker's prize and the other's prize in every outcomes. This is the motivation for one of my two key axioms, *position-preserving independence.*

Consider the next example which will explain the other key axiom *equivalence.* Compare the right lottery $\big((\$10,-\$10),.5;(-\$10,\$10).5\big)$ in Figure 2 with another one $\big((\$10,\$10),.5;(-\$10,-\$10),.5\big)$ which arise from a coin toss in which $(\$10,\$10)$ is realized if heads comes up and $(-\$10,-\$10)$ is realized otherwise. Note that both people have the same marginal distribution in both lotteries, which yield \$10 and $-\$10$ with a probability .5. However, the decision maker will prefer $\big((\$10,\$10),.5;(-\$10,-\$10),.5\big)$, since the outcome is the same for both people in this lottery as opposed to the other one. Hence, this evidence shows two lotteries which have same marginal distributions are not necessarily indifferent. That is because, marginal distribution does not give information regarding a sign of differences in each outcome. Hence, in order for marginal distribution to give enough information for a decision maker to evaluate the lottery, signs of difference in each outcome must be non-opposite. This is the motivation for the other key axiom, *equivalence.*

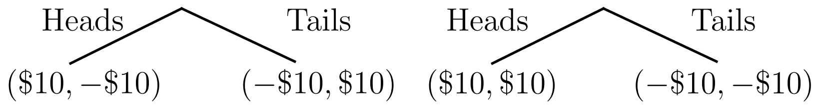

Figure 3:

This paper will show that these two key axioms in addition to the weak ordering and the continuity axioms are necessary and sufficient for the following representation of social preferences under uncertainty: Let I be a set of finite number of people including person 1, the decision maker. Consider a lottery f of which support is a

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[2] I have a plan of real experiments including this though experiments

subset of X^I. For $x \in X^I$, x_i is the prize for person i. Then,

$$
\begin{aligned}
U(f) = \quad & \delta \quad \Bigg(\sum_{x \in X^I} f(x) u(x_1) - \sum_{i \in I \setminus \{1\}} \alpha_i \max \Big\{ \sum_{x \in X^I} f(x)[u(x_i) - u(x_1)], 0 \Big\} \\
& \qquad\qquad - \sum_{i \in I \setminus \{1\}} \beta_i \max \Big\{ \sum_{x \in X^I} f(x)[u(x_1) - u(x_i)], 0 \Big\} \Bigg) \\
& +(1-\delta) \Bigg(\sum_{x \in X^I} f(x) u(x_1) - \sum_{i \in I \setminus \{1\}} \alpha_i \sum_{x \in X^I} f(x) \max \big\{ u(x_i) - u(x_1), 0 \big\} \\
& \qquad\qquad - \sum_{i \in I \setminus \{1\}} \beta_i \sum_{x \in X^I} f(x) \max \big\{ u(x_1) - u(x_i), 0 \big\} \Bigg).
\end{aligned}
\tag{2}
$$

The function u is a Bernoulli utility of asocial preference $\succsim^*$. That is, in my model, the decision maker is a standard expected utility maximizer in an asocial situation where there are no other people. The parameters α_i and β_i represent the psychological impact of inequality when the decision maker is worse off and better off than person i, respectively. The parameter $\delta_i \in [0, 1]$ shows the relative importance of inequality of opportunity between the decision maker and person i. In the first square bracket multiplied by δ_i, differences in payoffs are measured by the expected utilities. So the term shows the decision maker cares about equality of opportunity. On the other hand, in the second square bracket multiplied by $1-\delta_i$, relative payoffs are measured by each ex post utility. Hence the term shows the decision maker cares about equality of outcome. The representation includes two extreme cases: When $\delta_i = 1$, the decision maker cares only about the equality of opportunity. When $\delta_i = 0$, the decision maker cares only about the equality of outcome. Assuming independence implies $\delta_i = 0$ for all $i \in I$.

1.1 Applications to Experiments

In this section, as applications of the model, I will show that this model can explain consistently several seemingly different experimental results; efficiency, procedural fairness, altruism, spite as well as inequality aversion. Conventional researches use different models to explain these different results.

1.1.1 Equality versus Efficiency

Recently, Charness & Rabin (2002), Engelman & Strobel (2004, 2006), Andoreoni & Miller (2002) claim that total payoffs of the group, which they call "efficiency," has a stronger influence than inequality aversions. They report the results that in dictator games, a substantial number of subjects choose an "efficient," but unfair payoff. The most striking result by Charness & Rabin (2002) is that almost 50% of subjects prefer $(400, 400)$ to $(750, 375)$, where the first coordinate is the payoff to the dictator and the second coordinate is that for the receiver. They claim that it is impossible to explain the result based on other models of inequality aversion by Bolton &

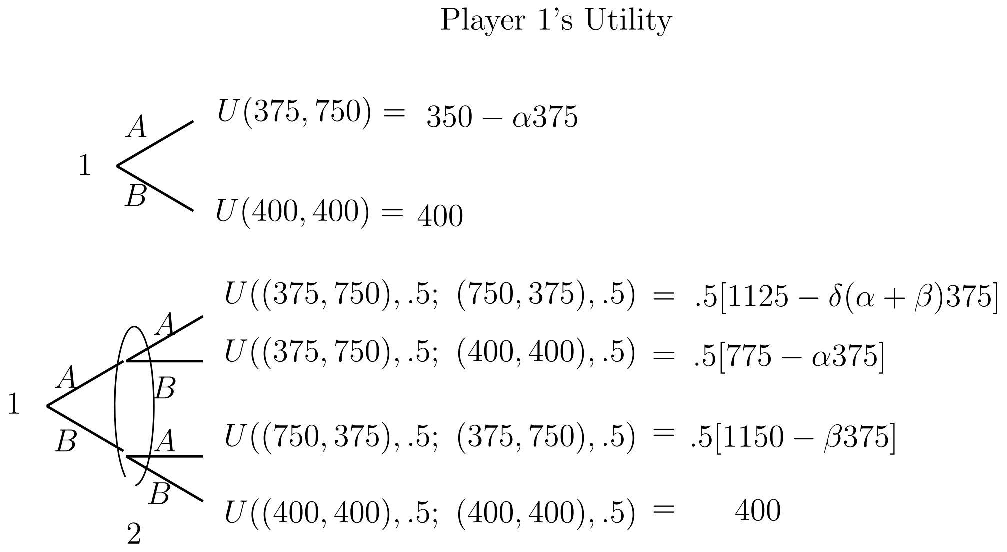
Ockenfels(2000) and Feher& Schmidt (1999). On the other hand, Bolton & Ockenfels (2006), Fehr et. al (2006), Guth et. al (2003), and Carlsson & Johansson-Stenman (2003), and other works follow similar procedure but report the opposite implication. In fact, in commenting to Engelman & Strobel (2004), Bolton & Ockenfels (2006) and Fehr. et al (2006) indicate that their replication of experiments indeed showed opposite tendencies of results. Bolton & Ockenfels (2006) suggests that procedual equity as used in Engelman & Strobel would make a difference. They explain that in Engelman & Strobel's experiments, a three person dictator game (one dictator and two receivers), subjects are required to make a decision as a dictator before their roles are determined. In other words, Bolton & Ockenfels (2006) claims that players had an equal opportunity to capture the efficiency gains. However, they suggest that the necessity to build a model of procedual equity, admitting that "social utility theory has been focused on allocation equity and has had little to say about procedural equity".

In this section, I will show that the model in the present paper can explain consistently the above experimental result in Charness & Rabin (2002) as well as inequality aversion. Especially, I will show that inequality averse subjects seem to prefer "efficient" payoffs to fair payoffs if a dictator game includes uncertainty of role. Charness & Rabin (2002) also employ the similar experimental procedure as Engelman & Strobel, in a two person dictator game. Charness & Rabin (2002) requires subject to make choice as a dictator before their roles are decided by flipping coin. Charness & Rabin (2002) requires that both subjects make their own decision on sharing before the actual roles of dictator and receiver are determined by a coin flip. The game tree which describes this game is the lower tree in the figure below, not the upper tree as Charness & Rabin assume. In the trees, An action A yields 375 point for the player and 750 point for the other if the player is chosen to be a dictator. An action B yields 400 point for both people equally if the player is chosen to be a dictator. If the player turns out to be a receiver, then the action does not affect any payoff.

For simplicity, assume that subjects are risk neutral and that there are two types of subjects, fair type and selfish type. The fair type players' parameters satisfy $\alpha \neq 0 \neq \beta$ and $\delta \neq 0$, or they are concerned about equality of opportunity. The selfish type players' parameters satisfy $\alpha = 0 = \beta$ or $\delta = 0$, or they are not concerned about equality of opportunity. Given the opponent action, the utility of the fair decision maker who choose the same action is the utility from a lottery which yields (x, y) and (y, x) with equal probability:

$$\begin{aligned} U\Big((x,y),.5;(y,x),.5\Big) &= \delta\Big[\tfrac{x+y}{2}\Big] + (1-\delta)\Big[\tfrac{x-\alpha[y-x]^{+}-\beta[x-y]^{+}}{2} + \tfrac{y-\alpha[x-y]^{+}-\beta[y-x]^{+}}{2}\Big] \\ &= \frac{1}{2}\Big[(x+y) - (1-\delta)(\alpha+\beta)|x-y|\Big] \\ &= \frac{1}{2}\Big[(\text{Efficiency}) - (1-\delta)(\text{Inequality Aversion})\Big]. \end{aligned}$$

This means that if subjects weigh equality of opportunity (δ is large), then subjects



$$U(375,750) = 350 - \alpha 375$$

$$U(400,400) = 400$$

$$U((375,750),.5;\ (750,375),.5) = .5[1125 - \delta(\alpha+\beta)375]$$

$$U((375,750),.5;\ (400,400),.5) = .5[775 - \alpha 375]$$

$$U((750,375),.5;\ (375,750),.5) = .5[1150 - \beta 375]$$

$$U((400,400),.5;\ (400,400),.5) = 400$$

Figure 4: Trees

seem to maximize efficiency rather than fairness. For example, it's easy to see that under parameters $\alpha = 0.5$, $\beta = 0.9$, $\sigma = 1.5$, if half of the subjects are fair, then in a Bayesian Nash equilibrium, fair type will choose $(375, 750)$ and selfish type will choose $(400, 400)$. Moreover, since α is positive, fair type chooses $(400, 400)$ over $(375, 750)$ without role uncertainty. [3]

1.1.2 Procedural Fairness

Bolton & Ockenfels (2006) suggests that "social utility theory has been focused on allocation equity and has had little to say about procedural equity". Bolton & Brands & Ockenfels (2005) carry out versions of dictator games. In one experiment, allocations are chosen by randomness, not by a dictator. If a receiver accepts an offer, the allocation realizes. Otherwise, both gets nothing. Bolton.et.al obtain result that the rejection rate of unfair allocation is higher if the allocation is chosen by unfair probability than if the allocation is chosen by fair probability. It is easy to see the model of this paper can explain the result[4].

1.1.3 Altruism and Spite

If parameters α and β are positive, then the model (2)shows inequality aversion. If parameter α is negative, then he can be interpreted as altruistic because his utility

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[3] In dictator games by Andoreoni & Miller (2002), all subjects are required to make decision as both dictator and receiver and they obtain sum of payoffs as dictator and receiver. So, Andoreoni & Miller (2002)'s experiment does not include role uncertainty. However, if subjects consider two separate games as dictator and receiver to be one game, it is easy to see that similar argument still holds because uncertainty as to types of opponents still exists.

[4] Trautman (2005) considers similar models

increases when others have better goods than him. If parameter β is negative, then he can be interpreted as spite because his utility increases when others have worse goods than him.

1.2 Alternative Interpretation from Multi-Period Consumption

In this section, I explain that the model used in this paper can be generalized into an alternate model of Gilboa (1989) and Shalev (1980) by a small change in the definition of position preserving independence.

In Gilboa (1989) and Shalev (1980), a decision maker are assumed to have a preference $\succsim$ over streams of lotteries $(f_1, f_2, \ldots, f_n)$, where each f_i is a lottery at period i. They assume also that the decision maker's preference $\succsim^*$ over one-period lotteries satisfies the expected utilities. Gilboa (1989) considers a simple example of four-period consumption. At each period, his payoff may be either high (H) or low (L). He suggests that if there are certain "costs of adjustment" incurred by any change in the payoff level then it seems plausible that the decision maker's preference relation $\succsim$ would satisfy

$$(H, H, L, L) \sim (L, L, H, H) \succ (H, L, H, L) \sim (L, H, L, H). \tag{3}$$

Based on the Gilboa's model, Shalev (1990) obtains the following representation:

$$\begin{aligned} U(f) = \sum_{x_1 \in X} f(x_1)u(x_1) + \sum_{t=2}^{n} \Big[\gamma^+(t) \max \Big\{ \sum_{x_t \in X} f(x_t)u(x_t) - \sum_{x_{t-1} \in X} f(x_{t-1})u(x_{t-1}), 0 \Big\} \\ + \gamma^-(t) \min \Big\{ \sum_{x_t \in X} f(x_t)u(x_t) - \sum_{x_{t-1} \in X} f(x_{t-1})u(x_{t-1}), 0 \Big\} \Big], \end{aligned} \tag{4}$$

where the function u is a Bernoulli utility function of preference $\succsim^*$ on one-period lotteries. Note that if $\gamma_+ + \gamma_- = 1$ and $\gamma_- > \gamma_+$, then the above representation (4) represents the preference (3).

However, the above representation (4) assume that there is no correlation in outcome across periods and people do not care about the correlations. Consider two lotteries $((H, H, L, L), .5; (L, L, H, H), .5)$ and $((H, L, H, L), .5; (L, H, L, H), .5)$. It seems plausible that the decision maker who has the preference (3) strictly prefer the first lottery to the second one. Note that marginal distribution of the lotteries are the same in each period. Since they assume decision makers have preference on sequence of marginal distribution, the representation (4) always suggests that both lotteries should be indifferent under any parameters.

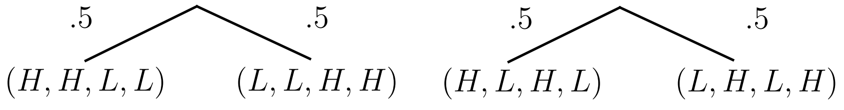

Figure 5:

However, for example, stock dividends are correlated across periods and people do care about the correlation. Based on this motivation, a natural generalization of the representation (4) can be expressed as follows:

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\begin{aligned}
U(f) = \quad \delta \ \Bigg(& \sum_{x \in X^T} f(x)u(x_1) + \sum_{t=2}^{n} \gamma^+(t) \max \Big\{ \sum_{x \in X^T} f(x)[u(x_t) - u(x_{t-1})], 0 \Big\} \\
& \qquad + \sum_{t=2}^{n} \gamma^-(t) \min \Big\{ \sum_{x \in X^T} f(x)[u(x_t) - u(x_{t-1})], 0 \Big\} \\
+(1-\delta) \Bigg(& \sum_{x \in X^T} f(x)u(x_1) + \sum_{t=2}^{n} \gamma^+(t) \sum_{x \in X^T} f(x) \max \big\{ u(x_t) - u(x_{t-1}), 0 \big\} \\
& \qquad + \sum_{t=2}^{n} \gamma^-(t) \sum_{x \in X^T} f(x) \min \big\{ u(x_t) - u(x_{t-1}), 0 \big\} \Bigg).
\end{aligned} \tag{5}
$$

The first term shows that the decision maker is cared about the correlation across marginal distributions in each period. The second term shows that the decision maker is cared about correlation across outcomes in each period. This representation (5) shows that if $\gamma_- > \gamma_+$, then the decision maker who has a preference (3) will prefer $((H,H,L,L),.5;(L,L,H,H),.5)$ to $((H,L,H,L),.5;(L,H,L,H),.5)$.

Note that this representation is very similar to the representation (2). The only difference is the reference point: The reference point in (2) is the a-social utility of the decision maker, $\sum_{x \in X^I} f(x)u(x_1)$. On the other hand, the reference point in (5) is the utility of one period past,or $\sum_{x_{t-1} \in X} f_{t-1}(x_{t-1})u(x_{t-1})$. Later, I will explain that this difference comes from a small difference between definitions of two weak versions of independence axioms.

2 The Model

Let X be the set of prizes. I assume that X is non-empty and connected.[5] For some positive integer n, let $I = \{1, \ldots, n\}$ be the set of individuals. I assume that person denoted by 1 is the decision maker. I use $\mathcal{X}$ to denote X^I. Let $\mathcal{F}$ be the set of finite support distribution on $\mathcal{X}$. For all $i \in I$, I use x_i and f_i to denote an element of X and F, respectively. I also use x and f to denote an element of $\mathcal{X}$ and $\mathcal{F}$, respectively. For all $x \in \mathcal{X}$ and $i \in I$, let $x_i \in X$ be an i-th element of x. For all $f \in \mathcal{F}$ and $i \in I$, let $f_i \in F$ be a marginal distribution of i-th element of f, or marginal lottery of person i. For all $f \in \mathcal{F}$ and $x \in \mathcal{X}$, $f(x)$ means the probability that the pair x of prizes is realized. For all $f_1 \in F$ and $x_1 \in X$, $f_1(x_1)$ means the probability that the prize x_1 is realized.

For any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g$ denote the lottery $h \in \mathcal{F}$ such that for all $x \in \mathcal{X}$, $h(x) = \alpha f(x) + (1-\alpha)g(x)$. Let $\succsim$ be a binary relation on $\mathcal{F}$. I use $f \succ g$, *f is strictly preferred to g*, to denote $f \succsim g$ and not $g \succsim f$. I use $f \sim g$, *f is indifferent to g*, to denote $f \succsim g$ and $g \succsim f$. For all $f_1 \in F$, $\overline{f_1}$ denotes the

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[5] I use topology generated by the l^1 metric.

probability distribution such that for all $x \in \mathcal{X}$,

$$\overline{f_1}(x) = \begin{cases} f_1(x_1) & \text{if } x = (x_1, \dots, x_1) \text{ for some } x_1 \in \text{supp}(f_1), \\ 0 & \text{otherwise.} \end{cases}$$

By $\overline{f_1}$, everyone always gets the same prize and everyone's marginal lottery is f_1. In other words, $\overline{f_1}$ is a lottery which is equal both in opportunity and in outcome. Note that for all $f_1, g_1 \in F$ and $\alpha \in [0,1]$, $\overline{\alpha f_1 + (1-\alpha) g_1} = \alpha \overline{f_1} + (1-\alpha)\overline{g_1}$. For all $x \in \mathcal{X}$ and $x_1 \in X$, I use $\langle x \rangle$ and $\langle x_1 \rangle$ to denote a lottery $f \in \mathcal{F}$ such that $f(x) = 1$ and a lottery $f_1 \in F$ such that $f_1(x_1) = 1$, respectively. I assume that for all $f \in \mathcal{F}$, there exists $x_1, y_1 \in X$ such that $\langle x_1, \dots, x_1 \rangle \succ f \succ \langle y_1, \dots, y_1 \rangle$. This assumption implies there exist element $x^+, x^- \in X$ such that $\langle x^+, \dots, x^+ \rangle \succ \langle x^-, \dots, x^- \rangle$. Henceforth, I fix these x^+ and x^-.

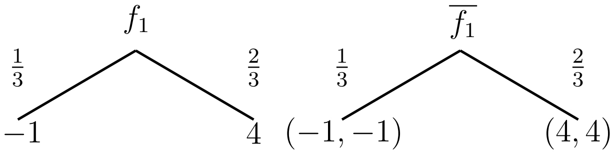

Figure 6: Example of f_1 and $\overline{f_1}$

I generate a preference $\succsim^*$ on F from $\succsim$ on $\mathcal{F}$ by $f_1 \succsim^* g_1 \Leftrightarrow \overline{f_1} \succsim \overline{g_1}$ for all $f_1, g_1 \in F$. $\succsim^*$ can be interpreted as the decision maker's selfish preference, or individual preference when the decision maker does not care about any other persons. That is because, if two lotteries are indifferent with respect to both equality of opportunity and equality of outcome, then the decision maker makes his choice based only on evaluation of his own marginal lottery.

2.1 Axioms

First, I introduce two standard axioms:

Weak Ordering (WO): $\succsim$ *is complete and transitive.*

Continuity (CT): *For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} | g \succsim f\}$ and $\{g \in \mathcal{F} | f \succsim g\}$ are closed.*

If $\succsim$ satisfies CT then, for all $f, g, h \in \mathcal{F}$ such that $f \succsim g \succsim h$ there exists a real number $\alpha \in [0,1]$ such that $g \sim \alpha f + (1-\alpha) h$.[6] By assumption, for any lottery

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[6] Choose any $f, g, h \in \mathcal{F}$ such that $f \succsim g \succsim h$. Define $g(\alpha) = \alpha f + (1-\alpha) h$, $B = \{\alpha \in [0,1] | g(\alpha) \succsim g\}$, and $W = \{\alpha \in [0,1] | g \succsim g(\alpha)\}$. Obviously, $1 \in B$ and $0 \in W$, so that both sets are non empty. Furthermore $[0,1] = B \cup W$. To show B is closed, choose any sequence $\{\alpha_n\}$ in B. Then $\{g(\alpha_n)\}$ is a sequence in $\{f \in \mathcal{F} | f \succsim g\}$.By CT, $\{f \in \mathcal{F} | f \succsim g\}$ is closed. Hence $g(\alpha_\infty) \in \{f \in \mathcal{F} | f \succsim g\}$. So by the continuity of g, $\alpha_\infty \in B$. Therefore, B is closed. W is also closed by the same reason. Hence by the connectedness of $[0,1]$, $B \cap W \neq \emptyset$. Choose any $\alpha \in B \cap W$. Then $g \sim g(\alpha) \equiv \alpha f + (1-\alpha) h$.

$f \in \mathcal{F}$, there exist $x_1, y_1 \in X$ such that $\langle x_1, \ldots, x_1 \rangle \succ f \succ \langle y_1, \ldots, y_1 \rangle$. Then CT shows that the existence of a lottery which is equal both in opportunity and in outcome and is indifferent to f.

Under the assumption that $\succsim$ satisfies WO,CT, and X is connected, for all $f_1 \in F$, there exists a certainty equivalent $x_1 \in X$ of f_1 such that $f_1 \sim^* \langle x_1 \rangle$ [7] and is denoted by $CE(f_1)$. Although for some $f_1 \in F$, there may exist $x, x' \in X$ such that $x \neq x'$ but $x \sim^* f_1 \sim^* x'$, when there is no risk of confusion, I use $CE(f_1)$ to denote arbitrary certainty equivalent of f_1.

The following three axioms are essential to derive the form of utility function. The first axiom is a version of comonotonic independence. However, it differs from comonotonic independence as defined in Schmeidler (1989) by using the index of people and comparing only the decision maker (person 1) and another person i rather than every pair of persons. In order to introduce the axiom, let me give you an example which violates normal independence. Consider two person case. It is no doubt that the other-regarding decision maker prefers the degenerate lottery $(0, 0)$ to $(0, -1)$, where $(0, 0)$ means both get nothing and $(0, -1)$ means the decision maker gets nothing and the other person get disutility -1. However, when these two lotteries are mixed with the degenerate lottery $(0, 1)$ with the probability $1/2$, respectively, the first lottery becomes $g \equiv (1/2)\langle 0, 0 \rangle + (1/2)\langle 0, 1 \rangle$ and the second becomes $f \equiv (1/2)\langle 0, -1 \rangle + (1/2)\langle 0, 1 \rangle$. A person who puts more weight on equality in opportunity than equality in outcome may prefer f to g. This is a violation of independence axiom. This preference reversal is caused by the fact that mixing a degenerate lottery $(0, 1)$, in which the decision maker is worse off in opportunity and outcome, offsets the inequality in opportunity of $(0, -1)$, in which the decision maker is better off in opportunity and outcome. Therefore, in order to evade this kind of violation, two lotteries which are mixed with each other must have non-opposite direction of inequality in opportunity.

Definition: For all $f, g \in \mathcal{F}$, f and g are said to be *position preserving* if f and g are comonotonic with respect to the marginal distribution of the first element, that is there is no $i \in I \setminus \{1\}$ such that,

$$f_1 \succ^* f_i \text{ and } g_i \succ^* g_1.$$

If two lotteries are position preserving, then for any person i, both lotteries have same directions of inequality in opportunity compared to the decision maker. I impose independence axiom within the set of pairwise position preserving lotteries.

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[7] By WO and CT, the definition shows $\succsim^*$ is a weak order satisfying the continuity. That is , for all $f_1 \in F$, $\{x \in X | \langle x_1 \rangle \succsim^* f_1\}$ and $\{x \in X | f_1 \succsim^* \langle x_1 \rangle\}$ are closed. Choose any $f_1 \in F$. By assumption, $\{x \in X | \langle x_1 \rangle \succsim^* f_1\} \neq \emptyset$ and $\{x \in X | f_1 \succsim^* \langle x_1 \rangle\} \neq \emptyset$. By definition $X = \{x \in X | \langle x_1 \rangle \succsim^* f_1\} \cup \{x \in X | f_1 \succsim^* \langle x_1 \rangle\}$. By the connectedness of X shows that $\{x \in X | \langle x_1 \rangle \succsim^* f_1\} \cap \{x \in X | f_1 \succsim^* \langle x_1 \rangle\} \neq \emptyset$. Choose any $x \in \{x \in X | \langle x_1 \rangle \succsim^* f_1\} \cap \{x \in X | f_1 \succsim^* \langle x_1 \rangle\}$. Then $f_1 \sim^* \langle x_1 \rangle$.

Opportunity Independence (OI): *For all* $\alpha \in [0,1]$ *and* $f, g, h \in \mathcal{F}$*, if* f, h *and* g, h *are position preserving or* $f_i \sim^* g_i$ *for all* $i \in I$ *then* $f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$*.*

Let $n = 2$ for brief explanation of the next axiom. Consider two lotteries f and g such that the decision maker is always better (worse) off than the other person in all realization of f and g. Then the inequality in outcome coincides with the difference in expected utilities of each person's marginal lottery. That is because there is no position reversal whichever prize may be realized. If each person's marginal distributions of lotteries f and g are indifferent, then f and g becomes indifferent with respect to not only inequality in opportunity but also inequality in outcome, so that f and g are indifferent.

Equivalence (EQ): *For all* $f, g \in \mathcal{F}$ *such that any elements of* $supp(f)$ *and* $supp(g)$ *are pairwise position preserving, if* $f_i \sim^* g_i$ *for all* $i \in I$ *then* $f \sim g$*.*

Note that EQ implies that for any $x, y \in \mathcal{X}$, if $\langle x_i \rangle \sim^* \langle y_i \rangle$ for all $i \in I$ then $\langle x \rangle \sim \langle y \rangle$. Hence the notation $[CE(f_i)]_{i \in I}$ is well defined. Define $x^0 = CE(1/2\langle x^+ \rangle + 1/2\langle x^- \rangle)$. For all $i \in I \setminus \{1\}$, define

$$b^i = \langle x^+, (x^0)_{-i} \rangle \text{ and } w^i = \langle x^-, (x^0)_{-i} \rangle.^{8}$$

That is for all $i \in I \setminus \{1\}$, b^i is a lottery where only person i is better off than the others. w^i is a lottery where only person i is worse off than the others. I need one more axiom just for the non-negativeness of the parameters.

Inequality Aversion (IA): *For all* $i \in I \setminus \{1\}$*,*
(i) $\langle x^0, \dots, x^0 \rangle \succ b^i$ *and* $\langle x^0, \dots, x^0 \rangle \succ w^i$*,*
(ii) $\langle x^0, \dots, x^0 \rangle \succsim \frac{1}{2} b^i + \frac{1}{2} w^i$*,*
(iii) $\frac{1}{2} b^i + \frac{1}{2} w^i \succsim \frac{1}{2}\langle x_1^i, \dots, x_1^i \rangle + \frac{1}{2}\langle y_1^i, \dots, y_1^i \rangle$*, where* $b^i \sim \langle x_1^i, \dots, x_1^i \rangle$ *and* $w^i \sim \langle y_1^i, \dots, y_1^i \rangle$ *for some* $x_1^i, y_1^i \in X$*.*

(i) shows that the decision maker does not prefer inequality whether he is better off or not. (ii) implies outcome inequality aversion because $(1/2)b^i + (1/2)e^i$ is unequal only in outcome. (iii) implies the opportunity inequality aversion. Although b^i and w^i are indifferent to $\langle x_1^i, \dots, x_1^i \rangle$ and $\langle y_1^i, \dots, y_1^i \rangle$, respectively, if the decision maker cares about equality of opportunity then $(1/2)b^i + (1/2)w^i$ is preferred to $(1/2)\langle x_1^i, \dots, x_1^i \rangle + (1/2)\langle y_1^i, \dots, y_1^i \rangle$. That is because, in $(1/2)b^i + (1/2)w^i$, inequalities of opportunity are offset by mixing non-position preserving two lotteries.

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[8] For any general element ν, μ and $i \in I$, let $(\nu, (\mu)_{-i})$ denote $(\underbrace{\mu, \dots, \mu}_{(i-1)\text{elements}}, \nu, \underbrace{\mu, \dots, \mu}_{(n-i)\text{elements}})$.

2.2 Main Theorem

Under the setting, I prove the following main theorem.

THEOREM: $\succsim$ *satisfies WO, CT, OI, OI, EQ, and IA if and only if there exist parameters* $(\alpha_i, \beta_i, \delta_i)_{i \in I \setminus \{1\}}$ *and vN-M utility function* u *such that*

$$\begin{aligned} U(f) \ = \ \textstyle\sum_{x \in \mathcal{X}} f(x) u(x_1) \ & - \ \textstyle\sum_{i \in I \setminus \{1\}} \delta_i \alpha_i \max \big\{ \textstyle\sum_{x \in \mathcal{X}} f(x)[u(x_i) - u(x_1)], 0 \} \\ & - \ \textstyle\sum_{i \in I \setminus \{1\}} \delta_i \beta_i \max \big\{ \textstyle\sum_{x \in \mathcal{X}} f(x)[u(x_1) - u(x_i)], 0 \} \\ & - \ \textstyle\sum_{i \in I \setminus \{1\}} (1 - \delta_i) \alpha_i \textstyle\sum_{x \in \mathcal{X}} f(x) \max \big\{ u(x_i) - u(x_1), 0 \big\} \\ & - \ \textstyle\sum_{i \in I \setminus \{1\}} (1 - \delta_i) \beta_i \textstyle\sum_{x \in \mathcal{X}} f(x) \max \big\{ u(x_1) - u(x_i), 0 \big\} \end{aligned}$$

represents $\succsim$ *and*
(i) $\alpha_i, \beta_i, \delta_i$ *are unique and* $\alpha_i > 0, \beta_i > 0,$ *and* $1 \geq \delta_i \geq 0$ *for all* $i \in I \setminus \{1\}$*,*
(ii) u *is a Bernoulli utility function associated with von Neumann-Morgenstern utility function representing* $\succsim^*$*,*
(ii) u *represents* $\succsim^*$*,*
(iii) U *and* u *are unique up to a joint positive affine transformation,*
(iv-a) $\alpha_i \geq \beta_i \Leftrightarrow b^i \lesssim w^i$ *for all* $i \in I \setminus \{1\}$*,*
(iv-b) $\alpha_i \geq \alpha_j \Leftrightarrow b^i \lesssim b^j$ *for all* $i, j \in I \setminus \{1\}$*,*
(iv-c) $\beta_i \geq \beta_j \Leftrightarrow w^i \lesssim w^j$ *for all* $i, j \in I \setminus \{1\}$*,*
(iv-d) $1 \geq \beta_i \Leftrightarrow f^- \lesssim w^i$ *for all* $i \in I \setminus \{1\}$*.*

The property (i) shows that these parameters are unique. Since α and β are positive, the decision maker does not like inequality. (ii) shows that if the decision maker is selfish, i.e., $\alpha = 0 = \beta$, then the decision maker's utility function becomes a vN-M utility function. (iii) means the following: Suppose that $U : \mathcal{F} \to \mathbb{R}$ defined by vN-M utility function u and parameters α, β, δ represents $\succsim$ on $\mathcal{F}$. Then $\tilde{U} : \mathcal{F} \to \mathbb{R}$ defined by vN-M utility function $\tilde{u}$ and the same parameters α, β, δ represents $\succsim$ on $\mathcal{F}$ if and only if there are scales $c \in \mathbb{R}_+$ and $d \in \mathbb{R}$ such that $\tilde{U}(f) = cU(f) + d$ for all $f \in \mathcal{F}$ and $\tilde{u}(f_1) = cu(f_1) + d$ for all $f_1 \in F$. (iv-a) shows the equivalent condition to the property that the decision maker suffers more from disadvantageous inequality than advantageous inequality. (iv-b) is on the comparison between disutilities when the decision maker is worse off than person i and j. (iv-c) is a counterpart property of (iv-b) when the decision maker is better off than person i. (iv-d) shows the condition of the limit of disutility when the decision maker is better than person i.

2.3 Corollaries

2.3.1 Weighted Sum Representation

I axiomatize the form (2) as a special case of the main theorem. Under IA (ii) and (iii), $\langle x^0, \ldots, x^0 \rangle \succsim (1/2) b^i + (1/2) w^i \succsim (1/2) \langle x_1^i, \ldots, x_1^i \rangle + (1/2) \langle y_1^i, \ldots, y_1^i \rangle$, where $b^i \sim \langle x_1^i, \ldots, x_1^i \rangle$ and $w^i \sim \langle y_1^i, \ldots, y_1^i \rangle$ for some $x_1^i, y_1^i \in X$. If the relative importance of inequality of outcome does not change depending on each person then $\succsim$ should satisfy the following axiom:

Constant Importance of Outcome (CIO): There exists $\lambda \in [0,1]$ such that for all $i \in I \setminus \{1\}$, $(1/2)b^i + (1/2)w^i \sim \lambda[(1/2)\langle x_1^i, \dots, x_1^i\rangle + (1/2)\langle y_1^i, \dots, y_1^i\rangle] + (1-\lambda)\langle x^0, \dots, x^0\rangle$, where $b^i \sim \langle x_1^i, \dots, x_1^i\rangle$ and $w^i \sim \langle y_1^i, \dots, y_1^i\rangle$ for some $x_1^i, y_1^i \in X$.

Corollary 1: *$\succsim$ satisfies WO, CT, OI, OI, EQ, IA , and CIO if and only if there exist parameters $(\alpha_i, \beta_i)_{i \in I\setminus\{1\}}$, δ and vN-M utility function u such that*

$$\begin{aligned} U(f) = \quad & \delta \Big[\textstyle\sum_{x\in\mathcal{X}} f(x)u(x_1) - \sum_{i\in I\setminus\{1\}} \alpha_i \max\big\{ \sum_{x\in X^I} f(x)[u(x_i) - u(x_1)], 0\big\} \\ & \qquad - \textstyle\sum_{i\in I\setminus\{1\}} \beta_i \max\big\{ \sum_{x\in X^I} f(x)[u(x_1) - u(x_i)], 0\big\}\Big] \\ & +(1-\delta)\Big[\textstyle\sum_{x\in\mathcal{X}} f(x)u(x_1) - \sum_{i\in I\setminus\{1\}} \alpha_i \sum_{x\in\mathcal{X}} f(x) \max\big\{u(x_i) - u(x_1), 0\big\} \\ & \qquad - \textstyle\sum_{i\in I\setminus\{1\}} \beta_i \sum_{x\in\mathcal{X}} f(x) \max\big\{u(x_1) - u(x_i), 0\big\} \Big] \end{aligned}$$

represents $\succsim$ and
(i-a) α_i, β_i are unique and $\alpha_i > 0, \beta_i > 0$ for all $i \in I \setminus \{1\}$,
(i-b) δ is unique and $1 \geq \delta \geq 0$,
(ii),(iii), and (iv) are the same.

2.3.2 Representations of Extreme Preferences

I axiomatize two extreme cases caring either equality of opportunity or equality of outcome as corollaries of the main theorem. First, consider the extreme preference caring only equality of opportunity. The preference must satisfy IA (ii) with equivalence, i.e., $(1/2)b^i + (1/2)w^i \sim \langle x^0, \dots, x^0\rangle$, for all $i \in I \setminus \{1\}$, because each person's marginal distributions of both lotteries are indifferent. Then the representation reduces into the following form:

Corollary 2: *$\succsim$ satisfies WO, CT, OI, OI, EQ, and IA (especially (ii) with equivalence) if and only if there exist parameters $(\alpha_i, \beta_i)_{i\in I}$ and a vN-M utility function u such that*

$$\begin{aligned} U(f) \;=\; \textstyle\sum_{x\in\mathcal{X}} f(x)u(x_1) \;&-\; \textstyle\sum_{i\in I\setminus\{1\}} \alpha_i \max\Big\{ \sum_{x\in\mathcal{X}} f(x)[u(x_i) - u(x_1)], 0\Big\} \\ &-\; \textstyle\sum_{i\in I\setminus\{1\}} \beta_i \max\Big\{ \sum_{x\in\mathcal{X}} f(x)[u(x_1) - u(x_i)], 0\Big\} \end{aligned}$$

represents $\succsim$ and (i), (ii), (iii), and (iv) are the same.

Next consider the other extreme preference caring only equality of outcome. The preference must satisfy IA (iii) with equivalence, i.e., for all $i \in I \setminus \{1\}$, $(1/2)b^i + (1/2)w^i \sim (1/2)\langle x_1^i, \dots, x_1^i\rangle + (1/2)\langle y_1^i, \dots, y_1^i\rangle$, because $b^i \sim \langle x_1^i, \dots, x_1^i\rangle$ and $w^i \sim$

$\langle y_1^i, \ldots, y_1^i \rangle$. Then the representation reduces into the following form:

Corollary 3: $\succsim$ *satisfies WO, CT, OI, OI, EQ, and IA (especially (iii) with equivalence) if and only if there exist parameters* $(\alpha_i, \beta_i)_{i \in I}$ *and a vN-M utility function* u *such that*

$$\begin{aligned} U(f) &= \textstyle\sum_{x \in \mathcal{X}} f(x) u(x_1) &- \textstyle\sum_{i \in I \setminus \{1\}} \alpha_i \textstyle\sum_{x \in \mathcal{X}} f(x) \max \Big\{ u(x_i) - u(x_1), 0 \Big\} \\ & &- \textstyle\sum_{i \in I \setminus \{1\}} \beta_i \textstyle\sum_{x \in \mathcal{X}} f(x) \max \Big\{ u(x_1) - u(x_i), 0 \Big\} \end{aligned}$$

represents $\succsim$ *and (i), (ii), (iii), and (iv) are the same.*

3 Sketch of the Proof

The main steps in the proof of the theorem are the following: First, I derive a vN-M utility function $u : F \to \mathbb{R}$ representing $\succsim^*$ and utility function U representing $\succsim$. Then I show the affinity of U over the set of position preserving lotteries. Next, using the affinity and defining parameters α_i, β_i, and γ_i, I prove a relatively concise representation of $\succsim$ with u and the parameters. Finally, defining new parameter δ_i from α_i, β_i, and γ_i, I show that $\succsim$ is represented by u and α_i, β_i, and δ_i.

I begin with deriving u and U. WO,CT, and the definition of $\succsim^*$ shows $\succsim^*$ satisfies the expected utility axioms. Hence there exists a vN-M utility function u representing $\succsim^*$. Choose a utility function u such that $u(x^+) = 1$ and $u(x^-) = -1$. By definition, u is unique and $u(x^0) = 0$. Define

$$\mathcal{F}^* = \{\overline{f_1} \in \mathcal{F} | f_1 \in F\}.$$

Since $\succsim$ satisfies independence axiom within $\mathcal{F}^*$, there exists vN-M utility function U^* representing $\succsim$ restricted to $\mathcal{F}^*$ such that $U^*(\langle x^+, \ldots, x^+ \rangle) = 1$ and $U^*(\langle x^-, \ldots, x^- \rangle) = -1$. By definition, U^* is unique and $U^*(\langle x^0, \ldots, x^0 \rangle) = 0$. Choose any $f \in \mathcal{F}$. By CT, there exist $x_1, y_1 \in X$ and $\alpha \in [0, 1]$ such that $f \sim \alpha \langle x_1, \ldots, x_1 \rangle + (1 - \alpha) \langle y_1, \ldots, y_1 \rangle$. Define

$$U(f) = U^*\big(\alpha \langle x_1, \ldots, x_1 \rangle + (1 - \alpha) \langle y_1, \ldots, y_1 \rangle\big).$$

Obviously, U is well defined. U is unique because of the uniqueness of U^*. It can be shown that $U(\overline{f_1}) = u(f_1)$ for all $f_1 \in F$ and if f and g are position preserving then $U(\alpha f + (1 - \alpha) g) = \alpha U(f) + (1 - \alpha) U(g)$ for all $\alpha \in]0, 1[$.

Henceforth, I fix this u and U. For all $i \in I \setminus \{1\}$, define

$$\alpha_i = -U(b^i), \quad \beta_i = -U(w^i), \quad \gamma_i = -2U\Big(\frac{1}{2} b^i + \frac{1}{2} w^i\Big).$$

By IA, it can be shown that all parameters are non-negative. Since b^i is a degenerate lottery in which only person i is better off, α_i is the amount of disutility from inequality when person i is better off in opportunity and in outcome. Similarly, β_i is the amount of disutility from inequality when person i is worse off in opportunity and in outcome. A lottery $(1/2)b^i + (1/2)w^i$ is equal in opportunity but unequal in outcome. Hence γ_i is the sum of disutility when person i is better off or worse off in outcome. Choose any $f \in \mathcal{F}$. For all $i \in I \setminus \{1\}$, define

$$B_i = \textstyle\sum_{x \in \mathcal{X}} f(x) \max\{u(x_i) - u(x_1), 0\}, \quad W_i = \textstyle\sum_{x \in \mathcal{X}} f(x) \max\{u(x_1) - u(x_i), 0\}.$$

Let $n = 2$ for simplicity. B_2 is the amount of inequalities in outcome when person 2 is better off than the decision maker. Similarly, W_2 is the amount of inequalities in outcome when person 2 is worse off. Since $B_2 - W_2 = u(f_2) - u(f_1)$, $B_2 - W_2$ coincides with the amount of inequality in opportunity. Therefore, if $B_2 \geq W_2$ then the decision maker suffers from disadvantageous inequality both in opportunity and in outcome by $B_2 - W_2$, advantageous inequality in outcome by W_2, and disadvantageous inequality in outcome by $W_2 (= B_2 - (B_2 - W_2))$.

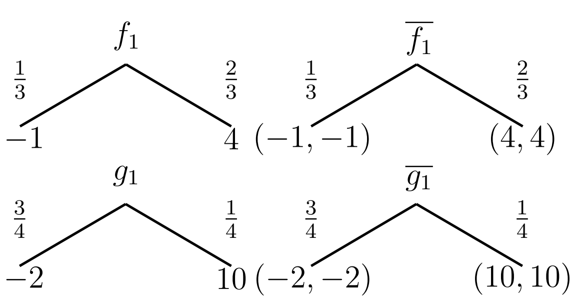

Figure 7: Example of f_1 and $\overline{f_1}$

Similarly, if $W_2 \geq B_2$ then the decision maker suffers from advantageous inequality both in opportunity and in outcome by $W_2 - B_2$, advantageous inequality in outcome by $B_2 (= W_2 - (W_2 - B_2))$, and disadvantageous inequality in outcome by B_2. Therefore, it can be shown that his preference is represented by the following form:

$$U(f) = u(f_1) - \alpha_2 \max\{B_2 - W_2, 0\} - \beta_2 \max\{W_2 - B_2, 0\} - \gamma_2 \min\{B_2, W_2\}. \tag{6}$$

The second and the third terms of the right hand side show disutilities from inequalities both in opportunity and in outcome when the decision maker is worse off or better off, respectively. The last term shows disutilities from inequalities in outcome. In order to explain the reason why the axioms implies the above form especially the second and the third terms, assume that the decision maker does not care about equality of outcome. Then it holds that $\gamma_2 = 0$ because $(1/2)b^2 + (1/2)w^2$ becomes indifferent to $\langle x^0, \ldots, x^0 \rangle$. Hence the form (6) reduces into: $U(f) = u(f_1) - \alpha_2 \max\{u(f_2) - u(f_1), 0\} - \beta_2 \max\{u(f_1) - u(f_2), 0\}$. Key property of the special form is that the affinity of U is established over the set of pairwise position preserving lotteries. This property is directly follows from OI which impose independence within the set of pairwise position preserving lotteries.

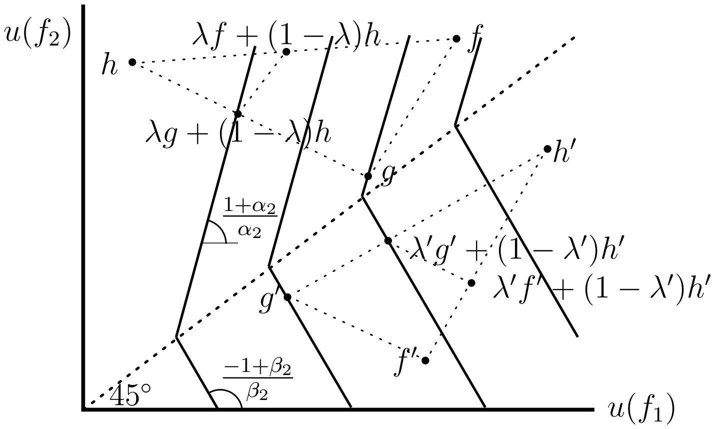

Figure 8: Utility Function Representing a Preference Caring only Equality of Opportunity: The affinity is established above and below the 45 degree line, respectively.

For all $i \in I \setminus \{1\}$, define

$$1 - \delta_i = \begin{cases} \dfrac{\gamma_i}{\alpha_i + \beta_i} & \text{if } \alpha_i + \beta_i = 0, \\ 0 & \text{if } \alpha_i + \beta_i \neq 0. \end{cases}$$

Note that since $\alpha_i + \beta_i \geq \gamma_i \geq 0$, if $\alpha_i + \beta_i = 0$ then $\gamma_i = 0$. Since γ_i is the amount of disutility in outcome and $\alpha_i + \beta_i$ is the amount of disutility both in opportunity and in outcome, the ratio δ_i shows the relative importance of inequality in outcome to inequalities both in opportunity and in outcome. With the parameter, (6) changes into the following form:[9]

$$\begin{aligned} U(f) = u(f_1) - \delta_2 \big[\alpha_2 \max\big\{B_2 - W_2, 0\big\} + \beta_2 \max\big\{W_2 - B_2, 0\big\}\big] \\ -(1-\delta_2)[\alpha_2 B_2 + \beta_2 W_2]. \end{aligned} \tag{7}$$

The definition of B_i and W_i and the affinity of u shows (7) is equivalent to (2) with $n = 2$. Properties (iv) are easily obtained by the definitions of the parameters in Step 10 in the appendix.

By the main theorem, three corollaries are easily proved. CIO shows δ is constant, so that Corollary 1 holds. IA (ii) with equivalence directly implies $\gamma_i = \alpha_i + \beta_i$, so that $\delta_i = 1$ for all $i \in I \setminus \{1\}$. Hence Corollary 2 is established. IA (iii) with equivalence shows $\gamma_i = 0$, so that $\delta_i = 0$ for all $i \in I \setminus \{1\}$. Therefore Corollary 3 is established.

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[9]

$$\begin{aligned} &\alpha_2 \max\{B_2 - W_2, 0\} + \beta_2 \max\{W_2 - B_2, 0\} + \gamma_2 \min\{B_2, W_2\} \\ &= \alpha_2 \max\{B_2 - W_2, 0\} + \beta_2 \max\{W_2 - B_2, 0\} + \delta_2(\alpha_2 + \beta_2) \min\{B_2, W_2\} \\ &= \delta_2(\alpha_2 \max\big\{B_2 - W_2, 0\big\} + \beta_2 \max\big\{W_2 - B_2, 0\big\}) + (1-\delta_2)\big(\alpha_2 \max\{B_2 - W_2, 0\} + \beta_2 \max\{W_2 - B_2, 0\} + (\alpha_2 + \beta_2) \min\{B_2, W_2\}\big) \\ &= \delta_2(\alpha_2 \max\big\{B_2 - W_2, 0\big\} + \beta_2 \max\big\{W_2 - B_2, 0\big\}) + (1-\delta_2)(\alpha_2 B_2 + \beta_2 W_2). \end{aligned}$$

Appendix

A Proof of Lemmas

In the appendix, for all $x_1 \in X$, I use $\overline{\langle x_1 \rangle}$ to denote $\langle x_1, \dots, x_1 \rangle$. I also use f^+, f^0, and $f^- \in \mathcal{F}$ to denote $\langle x^+, \dots, x^+ \rangle$, $\langle x^0, \dots, x^0 \rangle$, and $\langle x^-, \dots, x^- \rangle$, respectively.

Lemma 1:*Suppose that* $\succsim$ *satisfies WO,CT, OI, and OI. There exists a vN-M utility function* $u : F \to \mathbb{R}$ *representing* $\succsim^*$ *and a utility function* $U : \mathcal{F} \to \mathbb{R}$ *representing* $\succsim$ *such that*
(i) $u(x^+) = 1, u(x^0) = 0,$ *and* $u(x^-) = -1$,
(ii) $U(\overline{f_1}) = u(f_1)$ *for all* $f_1 \in F$,
(iii) u *is affine and unique,*
(iv) U *is unique,*
(v) for all $f, g \in \mathcal{F}$ *and* $\alpha \in]0,1[$*, if* f *and* g *are position preserving then* $U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$.

Proof of Lemma 1:
Step 1: There exists a unique vN-M utility function $u : F \to \mathbb{R}$ representing $\succsim^*$ such that $u(x^+) = 1, u(x^0) = 0,$ and $u(x^-) = -1$.
Proof of Step 1: By WO and CT, the definition shows $\succsim^*$ is a weak order satisfying the continuity. In addition, for all $f_1, g_1, h_1 \in F$ and $\alpha \in]0,1[$,

$$\begin{array}{rcll} f_1 \succsim^* g_1 & \Leftrightarrow & \overline{f_1} \succsim \overline{g_1} & (\because \text{Definition}) \\ & \Leftrightarrow & \alpha\overline{f_1} + (1-\alpha)\overline{h_1} \succsim \alpha\overline{g_1} + (1-\alpha)\overline{h_1} & (\because \text{OI}) \\ & \Leftrightarrow & \overline{\alpha f_1 + (1-\alpha)h_1} \succsim \overline{\alpha g_1 + (1-\alpha)h_1} & \\ & \Leftrightarrow & \alpha f_1 + (1-\alpha)h_1 \succsim^* \alpha g_1 + (1-\alpha)h_1. & (\because \text{Definition}) \end{array}$$

Hence $\succsim^*$ satisfies the independence axiom. The expected utility theorem shows that there exists a vN-M utility functions $u : F \to \mathbb{R}$ representing the preference relation $\succsim^*$, which is affine and unique up to a positive affine transformation. Let $u(x^+) = 1$ and $u(x^-) = -1$. Then u is unique and $u(x^0) = 0$, by the affinity of u. □

Define

$$\mathcal{F}^* = \{\overline{f_1} \in \mathcal{F} | f_1 \in F\}.$$

Step 2: There exists a unique vN-M utility function U^* representing $\succsim$ restricted to $\mathcal{F}^*$ such that $U^*(\overline{f_1}) = u(f_1)$ for all $f_1 \in F$.
Proof of Step 2: By WO,CT,OI, $\succsim$ restricted to $\mathcal{F}^*$ satisfies the expected utility axioms. Choose a vN-M utility function U^* such that $U^*(f^+) = 1$ and $U^*(f^-) = -1$. Then U^* is unique.

Choose any $f_1 \in F$ to show $U^*(\overline{f_1}) = u(f_1)$.
Case 1: $f^- \succsim \overline{f_1}$. Then by CT, there exists $\alpha \in [0,1]$ such that $f^- \sim \alpha f^+ + (1-\alpha)\overline{f_1}$. By definition, $x^- \sim^* \alpha x^+ + (1-\alpha) f_1$. Hence $U^*(\overline{f_1}) = \dfrac{-1-\alpha}{1-\alpha} = u(f_1)$.
Case 2: $f^+ \succsim \overline{f_1} \succsim f^-$. By the same way as Case 1.
Case 3: $\overline{f_1} \succsim f^+$. By the same way as Case 1. □
Step 3: There exists a unique utility function U representing $\succsim$.
Proof of Step 3: Choose any $f \in \mathcal{F}$. By assumption, there exist $x_1, y_1 \in X$ such that

$\overline{\langle x_1\rangle} \succsim f \succsim \overline{\langle y_1\rangle}$. By the continuity, there exists $\alpha \in [0,1]$ such that $f \sim \alpha\overline{\langle x_1\rangle}+(1-\alpha)\overline{\langle y_1\rangle}$. Define

$$U(f) = U^*(\alpha\overline{\langle x_1\rangle} + (1-\alpha)\overline{\langle y_1\rangle}).$$

Obviously, U is well defined and U represents $\succsim$. Uniqueness of U follows from that of U^*. □

Step 4: $U(\overline{f_1}) = u(f_1)$ for all $f_1 \in F$.

Proof of Step 4: For all $f_1 \in F$, $U(\overline{f_1}) = U^*(\overline{f_1}) = u(f_1)$. □

Step 5: For all $f, g \in \mathcal{F}$ and $\alpha \in]0,1[$, if f and g are position preserving then $U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$.

Proof of Step 5: Choose any $\alpha \in]0,1[$ and any $f, g \in \mathcal{F}$ such that f, g are position preserving. Then by CT, there exists $x_1, y_1 \in X$, $\beta, \gamma \in [0,1]$ such that $f \sim \beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}$ and $g \sim \gamma\overline{\langle x_1\rangle}+(1-\gamma)\overline{\langle y_1\rangle}$. Since f, g, and $\beta\overline{\langle x_1\rangle}+(1-\beta)\overline{\langle y_1\rangle}$ are pairwise position preserving, $\alpha f+(1-\alpha)g \sim \alpha[\beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}] + (1-\alpha)g$. Similarly, since $g, \beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}$, and $\gamma\overline{\langle x_1\rangle} + (1-\gamma)\overline{\langle y_1\rangle}$ are pairwise position preserving, $\alpha[\beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}] + (1-\alpha)g \sim \alpha[\beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}] + (1-\alpha)[\gamma\overline{\langle x_1\rangle}+(1-\gamma)\overline{\langle y_1\rangle}]$. Therefore, $\alpha f+(1-\alpha)g \sim \alpha[\beta\overline{\langle x_1\rangle}+(1-\beta)\overline{\langle y_1\rangle}]+(1-\alpha)[\gamma\overline{\langle x_1\rangle}+(1-\gamma)\overline{\langle y_1\rangle}]$. Hence

$$\begin{aligned} U(\alpha f + (1-\alpha)g) &= U^*(\alpha[\beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}] + (1-\alpha)[\gamma\overline{\langle x_1\rangle} + (1-\gamma)\overline{\langle y_1\rangle}]) \\ &= \alpha U^*(\beta\overline{\langle x_1\rangle} + (1-\beta)\overline{\langle y_1\rangle}) + (1-\alpha)U^*(\gamma\overline{\langle x_1\rangle} + (1-\gamma)\overline{\langle y_1\rangle}) \\ &= \alpha U(f) + (1-\alpha)U(g). \end{aligned}$$

□

■

Henceforth, by the end of the proof of main theorem, I fix these u and U.

Lemma 2: *Suppose that* $\succsim$ *satisfies CT and EQ.*

(i) Let $f \in \mathcal{F}$. *If* x *and* y *are position preserving for all* $x, y \in$ *supp*(f) *then* $f \sim [CE(f_i)]_{i\in I}$.

(ii) for all $x, y \in \mathcal{X}$, *if* $x_i \sim^* y_i$ *for all* $i \in I$ *then* $\langle x\rangle \sim \langle y\rangle$.

Proof of Lemma 2: Choose any f such that any elements of supp(f) are pairwise position preserving to show (i). Then for all $x, y \in$ supp(f), $[CE(f_i)]_{i\in I}$, x, and y are pairwise position preserving. Since, obviously, $f_i \sim^* CE(f_i)$ for all $i \in I$, EQ shows that $f \sim [CE(f_i)]_{i\in I}$.

Choose any $x, y \in \mathcal{X}$ such that $x_i \sim^* y_i$ for all $i \in I$ to show (ii). Obviously $\langle x\rangle$ and $\langle y\rangle$ are position preserving, so EQ shows $\langle x\rangle \sim \langle y\rangle$. ■

For all $f \in \mathcal{F}$ and $i \in I \setminus \{1\}$, define

$$B^i(f) = \{x \in \text{supp}(f) | x_1 \prec^* x_i\},\ W^i(f) = \{x \in \text{supp}(f) | x_1 \succ^* x_i\},\ F^i(f) = \text{supp}(f) \setminus [B^i(f) \cup W^i(f)].$$

Lemma 3: *Suppose that* $\succsim$ *satisfies WO, OI, and EQ. For all* $s \in \{1, \dots, m\}$, *let* $f^s, g^s \in \mathcal{F}$ *and* $\alpha_s \geq 0$ *such that* $\sum_{s=1}^m \alpha_s = 1$. *If supp*$(f^s)$ *and supp*(g^s) *are position preserving and* $f_i^s \sim^* g_i^s$ *for all* $s \in \{1, \dots, m\}$ *and* $i \in I$ *then* $\sum_{s=1}^m \alpha_s f^s \sim \sum_{s=1}^m \alpha_s g^s$.

Proof of Lemma 3: Since supp(f^s) and supp(g^s) are position preserving and $f_i^s \sim^* g_i^s$ for all $i \in I$, EQ shows that $f^s \sim g^s$ for all $s \in \{1, \dots, m\}$.

Since $f_i^1 \sim^* g_i^1$ for all $i \in I$ and $f^1 \sim g^1$, OI shows $\dfrac{\alpha_1}{\alpha_1+\alpha_2}f^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}f^2 \sim \dfrac{\alpha_1}{\alpha_1+\alpha_2}g^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}f^2$. Since $f_i^2 \sim^* g_i^2$ for all $i \in I$ and $f^2 \sim g^2$, OI again shows $\dfrac{\alpha_1}{\alpha_1+\alpha_2}g^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}f^2 \sim$

$\dfrac{\alpha_1}{\alpha_1+\alpha_2}g^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}g^2$. Therefore, $f' \equiv \dfrac{\alpha_1}{\alpha_1+\alpha_2}f^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}f^2 \sim \dfrac{\alpha_1}{\alpha_1+\alpha_2}g^1 + \dfrac{\alpha_2}{\alpha_1+\alpha_2}g^2 \equiv g'$. Furthermore, since $f_i^1 \sim^* g_i^1$ and $f_i^2 \sim^* g_i^2$, then $f_i' \sim^* g_i'$ for all $i \in I$. Therefore, by reiterating this calculation, I get $\sum_{s=1}^m \alpha_s f^s \sim \sum_{s=1}^m \alpha_s g^s$. ■

LEMMA 4: *Suppose that* $\succsim$ *satisfies WO, OI, and EQ. For all* $f \in \mathcal{F}$, *if for all* $x \in supp(f)$ *and* $i \in I$, *there exists* $z_i \in X$ *such that* $u(z_i) = u(x_i) - u(x_1)$ *then*

$$\frac{1}{n}f + \frac{n-1}{n}f^0 \sim \frac{1}{n}\sum_{x\in\mathcal{X}} f(x)\overline{\langle x_1\rangle} + \sum_{i\neq 1}\frac{1}{n}\sum_{x\in\mathcal{X}} f(x)\langle z_i, (x^0)_{-i}\rangle.$$

PROOF OF LEMMA 4: Let $\text{supp}(f) = \{x^1, \ldots, x^m\}$. Choose any $s \in \{1, \ldots, m\}$. Define $f^{x^s} = \dfrac{1}{n}\langle x^s\rangle + \dfrac{n-1}{n}f^0$ and $g^{x^s} = \dfrac{1}{n}\overline{\langle x_1^s\rangle} + \sum_{i\neq 1}\dfrac{1}{n}\langle z_i^s, (x^0)_{-i}\rangle$, where $u(z_i^s) = u(x_i^s) - u(x_1^s)$. For all $i \in I$, since $\dfrac{1}{n}u(x_i^s) + \dfrac{n-1}{n}u(x^0) = \dfrac{u(x_i^s)}{n} = \dfrac{1}{n}u(x_1^s) + \dfrac{1}{n}u(z_i^s) + \dfrac{n-1}{n}u(x^0)$, then $f_i^{x^s} \sim^* g_i^{x^s}$. Since $\langle x^s\rangle, f^0, \overline{\langle x_1^s\rangle}$, and $\{\langle z_i^s, (x^0)_{-i}\rangle | i \neq 1\}$ are pairwise position preserving, EQ shows $f^{x^s} \sim g^{x^s}$. Therefore, by Lemma 3,

$$\frac{1}{n}f + \frac{n-1}{n}f^0 = \sum_{x\in\mathcal{X}} f(x)f^x \sim \sum_{x\in\mathcal{X}} f(x)g^x = \frac{1}{n}\sum_{x\in\mathcal{X}} f(x)\overline{\langle x_1\rangle} + \sum_{i\neq 1}\frac{1}{n}\sum_{x\in\mathcal{X}} f(x)\langle z_i, (x^0)_{-i}\rangle.$$

■

LEMMA 5: *Suppose that* $\succsim$ *satisfies WO, OI, and EQ. Let* $f, g \in \mathcal{F}$. *If for all* $S \in \{B, F, W\}$ *and* $i \in I$ *and* $j \in I$,

$$\sum_{x\in S^i(f)} f(x)\langle x_j\rangle + \Big(1 - \sum_{x\in S^i(f)} f(x)\Big)\langle x_0\rangle \sim^* \sum_{y\in S^i(g)} g(y)\langle y_j\rangle + \Big(1 - \sum_{x\in S^i(g)} g(y)\Big)\langle x_0\rangle,$$

then $f \sim g$.

PROOF OF LEMMA 5: By EQ, for all $S \in \{B, F, W\}$ and $i \in I$,

$$\sum_{x\in S^i(f)} f(x)\langle x\rangle + \Big(1 - \sum_{x\in S^i(f)} f(x)\Big)\overline{\langle x_0\rangle} \sim \sum_{y\in S^i(g)} g(y)\langle y\rangle + \Big(1 - \sum_{x\in S^i(g)} g(y)\Big)\overline{\langle x_0\rangle},$$

Hence Lemma 3 shows

$$\begin{aligned}
\frac{1}{3}f + \frac{2}{3}\overline{\langle x_0\rangle} &= \frac{1}{(n-1)}\sum_{i\in I\setminus\{1\}}\frac{1}{3}\Big[\sum_{x\in\text{supp}(f)} f(x)\langle x\rangle + \Big(3 - \sum_{x\in\text{supp}(f)} f(x)\Big)\overline{\langle x_0\rangle}\Big] \\
&= \frac{1}{(n-1)}\sum_{i\in I\setminus\{1\}}\frac{1}{3}\sum_{S\in\{B,F,W\}}\Big[\sum_{x\in S^i(f)} f(x)\langle x\rangle + \Big(1 - \sum_{x\in S^i(f)} f(x)\Big)\overline{\langle x_0\rangle}\Big] \\
&\sim \frac{1}{(n-1)}\sum_{i\in I\setminus\{1\}}\frac{1}{3}\sum_{S\in\{B,F,W\}}\Big[\sum_{y\in S^i(g)} g(y)\langle y\rangle + \Big(1 - \sum_{x\in S^i(g)} g(y)\Big)\overline{\langle x_0\rangle}\Big] \\
&= \frac{1}{3}g + \frac{2}{3}\overline{\langle x_0\rangle},
\end{aligned}$$

so that $f \sim g$. ■

LEMMA 6: *Suppose that* $\succsim$ *satisfies WO, CT, OI, and EQ. Let* $f \in \mathcal{F}$. $\exists M(f) \in \mathbb{Z}_+, \forall x \in$ *supp*$(f), \forall i \in I, \exists z_i^x \in X$ *such that*

$$\begin{cases}
\dfrac{1}{nM(f)}f + \dfrac{nM(f)-1}{nM(f)}f_0 & \sim \quad \dfrac{1}{n}\sum_{x\in\mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum_{i\in I\setminus\{1\}}\dfrac{1}{n}\sum_{x\in\mathcal{X}} f(x)\langle z_x^i, (x_0)_{-i}\rangle, \\
u(z_1^x) & = \quad \dfrac{1}{nM(f)}u(x_1), \\
u(z_i^x) & = \quad \dfrac{1}{nM(f)}\Big(u(x_i) - u(x_1)\Big).
\end{cases}$$

PROOF OF LEMMA 6: Choose any $f \in \mathcal{F}$.
STEP 1: For all $f \in \mathcal{F}$ and positive integer M,

$$\frac{1}{M}f + \frac{M-1}{M}f^0 \sim \sum_{x \in \mathcal{X}} f(x)\Big[\Big[CE\Big(\frac{1}{M}\langle x_i\rangle + \frac{M-1}{M}\langle x^0\rangle\Big)\Big]_{i \in I}\Big].$$

PROOF OF STEP 1: Choose any $f \in \mathcal{F}$ and positive integer M. Let $\text{supp}(f) = \{x^1, \ldots, x^m\}$. Choose any $x^s \in \text{supp}(f)$. Let $f^{x^s} = \frac{1}{M}\langle x^s\rangle + \frac{M-1}{M}f^0$. Since f^0 and $\langle x^s\rangle$ are position preserving, Lemma 2 (i) shows that $f^{x^s} \sim [CE(f_i^{x^s})]_{i \in I}$. Obviously, $f_i^{x^s} \sim^* CE(f_i^{x^s})$ for all $i \in I$. Therefore by Lemma 3,

$$\frac{1}{M}f+\frac{M-1}{M}f^0 = \sum_{x \in \mathcal{X}} f(x)f^x \sim \sum_{x \in \mathcal{X}} f(x)[[CE(f_i^x)]_{i \in I}] = \sum_{x \in \mathcal{X}} f(x)\Big[\Big[CE\Big(\frac{1}{M}\langle x_i\rangle+\frac{M-1}{M}\langle x^0\rangle\Big)\Big]_{i \in I}\Big].$$

□

For all $M \in \mathbb{Z}_{++}$ and $i \in I$, define

$$x_i^M = CE\Big(\frac{1}{M}\langle x_i\rangle + \frac{M-1}{M}\langle x_0\rangle\Big).$$

Under the assumption that X is connected, CT shows the existence of x_i^M.

STEP 2: $\exists M(f) \in \mathbb{Z}_+, \forall x \in \text{supp}(f), \exists z_x^i \in X$ such that

$$\begin{cases} u(z_1^x) & = & u(x_1^{M(f)}), \\ u(z_i^x) & = & u(x_i^{M(f)}) - u(x_1^{M(f)}) \text{ for all } i \in I \setminus \{1\}. \end{cases}$$

PROOF OF STEP 2: Take M large enough to hold $1 \geq u\big(x_1^M\big) = \frac{u(x_1)}{M} \geq -1$ and $1 \geq u\big(x_i^M\big) - u\big(x_1^M\big) = \frac{u(x_i) - u(x_1)}{M} \geq -1$ for all $x \in \text{supp}(f)$ and $i \in I \setminus \{1\}$. Define

$$z_1^x = \begin{cases} CE\left(\frac{u(x_1)}{M}\langle x_+\rangle + \Big(1 - \frac{u(x_1)}{M}\Big)\langle x_0\rangle\right) & \text{if } u(x_1) \geq 0, \\ CE\left(\frac{u(x_1)}{M}\langle x_-\rangle + \Big(1 - \frac{u(x_1)}{M}\Big)\langle x_0\rangle\right) & \text{if } u(x_1) \leq 0. \end{cases}$$

z_i^x can be defined by the same way. □

STEP 3: $\frac{1}{n}\sum_{x \in \mathcal{X}} f(x)x^{M(f)} + \frac{n-1}{n}f^0 \sim \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum_{i \in I\setminus\{1\}} \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\langle z_i^x, (x^0)_{-i}\rangle$.

PROOF OF STEP 3: By Step 2 and Lemma 4. □

STEP 4: $\frac{1}{nM(f)}f + \frac{nM(f)-1}{nM(f)}f^0 \sim \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum_{i \in I\setminus\{1\}} \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\langle z_i^x, (x^0)_{-i}\rangle$.

PROOF OF STEP 4:

$$\begin{aligned} \frac{1}{nM(f)}f + \frac{nM(f)-1}{nM(f)}f^0 &= \frac{1}{n}\Big(\frac{1}{M(f)}f + \frac{M(f)-1}{M(f)}f^0\Big) + \frac{n-1}{n}f^0 \\ &\sim \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)x^{M(f)} + \frac{n-1}{n}f^0 && (\because \text{STEP 1}) \\ &\sim \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum_{i \in I\setminus\{1\}} \frac{1}{n}\sum_{x \in \mathcal{X}} f(x)\langle z_i^x, (x^0)_{-i}\rangle. && (\because \text{STEP 3}) \end{aligned}$$

$\square$

Step 2 and Step 4 establish the result. $\blacksquare$

LEMMA 7: *Suppose that $\succsim$ satisfies WO,CT, OI, and EQ. For all $f, g \in \mathcal{F}$,*

$$\left[\begin{array}{l} \sum_{x\in\mathcal{X}} f(x)u(x_1) = \sum_{y\in\mathcal{X}} g(y)u(y_1), \\ \forall i \in I\setminus\{1\} \left[\begin{array}{l} \sum_{x\in B^i(f)} f(x)[u(x_i)-u(x_1)] = \sum_{y\in B^i(g)} g(y)[u(y_i)-u(y_1)], \\ \sum_{x\in W^i(f)} f(x)[u(x_i)-u(x_1)] = \sum_{y\in W^i(g)} g(y)[u(y_i)-u(y_1)] \end{array}\right] \end{array}\right] \Rightarrow f \sim g.$$

PROOF OF LEMMA 7: Choose any f, g. In Lemma 3, it is easy to show to take M in common for both f and g. That is because, it is enough to take $M^* = \min\{M(f), M(g)\}$ in Step 2 of Lemma 3. Therefore, $\exists M^* \in \mathbb{Z}_+, \forall x \in \text{supp}(f), \forall i \in I, \exists z_i^x \in X$ such that

$$\left\{\begin{array}{lcll} \dfrac{1}{nM^*}f + \dfrac{nM^*-1}{nM^*}f_0 & \sim & \dfrac{1}{n}\sum\limits_{x\in\mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum\limits_{i\in I\setminus\{1\}}\dfrac{1}{n}\sum\limits_{x\in\mathcal{X}} f(x)\langle z_x^i, (x_0)_{-i}\rangle, & (1) \\ u(z_1^x) & = & \dfrac{1}{nM^*}u(x_1), & (2) \\ u(z_i^x) & = & \dfrac{1}{nM^*}\Big(u(x_i)-u(x_1)\Big), & (3) \end{array}\right.$$

and $\forall y \in \text{supp}(g), \forall i \in I, \exists w_i^y \in X$ such that

$$\left\{\begin{array}{lcll} \dfrac{1}{nM^*}g + \dfrac{nM^*-1}{nM^*}f_0 & \sim & \dfrac{1}{n}\sum\limits_{y\in\mathcal{X}} g(y)\overline{\langle w_1^y\rangle} + \sum\limits_{i\in I\setminus\{1\}}\dfrac{1}{n}\sum\limits_{y\in\mathcal{X}} g(y)\langle w_i^y, (x_0)_{-i}\rangle, & (4) \\ u(w_1^y) & = & \dfrac{1}{nM^*}u(y_1), & (5) \\ u(w_i^y) & = & \dfrac{1}{nM^*}\Big(u(y_i)-u(y_1)\Big). & (6) \end{array}\right.$$

Define

$$\begin{array}{lcl} f' & \equiv & \dfrac{1}{n}\sum\limits_{x\in\mathcal{X}} f(x)\overline{\langle z_1^x\rangle} + \sum\limits_{i\in I\setminus\{1\}}\dfrac{1}{n}\sum\limits_{x\in\mathcal{X}} f(x)\langle z_x^i, (x_0)_{-i}\rangle, \\ g' & \equiv & \dfrac{1}{n}\sum\limits_{y\in\mathcal{X}} g(y)\overline{\langle w_1^y\rangle} + \sum\limits_{i\in I\setminus\{1\}}\dfrac{1}{n}\sum\limits_{y\in\mathcal{X}} g(y)\langle w_i^y, (x_0)_{-i}\rangle. \end{array}$$

Therefore, for all $i, j \in I$,

$$\begin{array}{rcll} \sum_{x'^i(f')} f'(x)u(x'_j) & = & \dfrac{1}{n}\sum_{x\in\mathcal{X}} f(x)u(z_1^x) & (\because \text{Definition}) \\ & = & \dfrac{1}{n^2M^*}\sum_{x\in\mathcal{X}} f(x)u(x_1) & (\because (2)) \\ & = & \dfrac{1}{n^2M^*}\sum_{y\in\mathcal{X}} g(y)u(y_1) & (\because \text{Assumption}) \\ & = & \dfrac{1}{n}\sum_{y\in\mathcal{X}} g(y)u(w_1^y) & (\because \text{Definition}) \\ & = & \sum_{y'^i(g')} g'(y)u(y'_j), & (\because (5)) \end{array}$$

so that $\sum\limits_{x'^i(f')} f'(x')\langle x'_j\rangle + \Big(1 - \sum\limits_{x'^i(f')} f'(x')\Big)\langle x_0\rangle \sim^* \sum\limits_{y'^i(g')} g'(y')\langle y'_j\rangle + \Big(1 - \sum\limits_{x'^i(g')} g'(y')\Big)\langle x_0\rangle$. For all $i \in I$ and $S \in \{B, W\}$,

$$\begin{array}{rcll} \sum_{x'^i(f')} f'(x')u(x'_i) & = & \dfrac{1}{n}\sum_{x\in\mathcal{X}} f(x)u(z_i^x) & (\because \text{Definition}) \\ & = & \dfrac{1}{n^2M^*}\sum_{x\in S^i(f)} f(x)[u(x_i)-u(x_1)] & (\because (3)) \\ & = & \dfrac{1}{n^2M^*}\sum_{y\in S^i(g)} g(y)[u(y_i)-u(y_1)] & (\because \text{Assumption}) \\ & = & \dfrac{1}{n}\sum_{y\in\mathcal{X}} g(y)u(w_i^y) & (\because \text{Definition}) \\ & = & \sum_{y'^i(g')} g'(y')u(y'_i), & (\because (6)) \end{array}$$

and for all $i, j \in I$ such that $i \neq j$,

$$\begin{array}{rcll} \sum_{x'^i(f')} f'(x')u(x'_j) & = & 0 & (\because \text{Definition of } f') \\ & = & \sum_{y'^i(g')} g'(y')u(y'_j). & (\because \text{Definition of } g') \end{array}$$

Hence for all $i, j \in I$ and $S \in \{B, W\}$, $\sum\limits_{x'^i(f')} f'(x')\langle x'_j\rangle + \Big(1 - \sum\limits_{x'^i(f')} f'(x')\Big)\langle x_0\rangle \sim^* \sum\limits_{y'^i(g')} g'(y')\langle y'_j\rangle + \Big(1 - \sum\limits_{x'^i(g')} g'(y')\Big)\langle x_0\rangle$. Therefore, by Lemma 5, $f' \sim g'$. Hence $f \sim g$. ■

B Proof of Main Theorem

PROOF OF THEOREM:
STEP 1: For all $\alpha_1, \ldots, \alpha_K \in [0,1]$ and all $f^1, \ldots, f^k \in \mathcal{F}$, if $\sum_{k=1}^K \alpha_k = 1$ and $\{f^k\}$ are pairwise position preserving then $U\Big(\sum_{k=1}^K \alpha_k f^k\Big) = \sum_{k=1}^K \alpha_k U(f^k)$.
PROOF OF STEP 1: By Lemma 1 (v). □

Define

$$\hat{\mathcal{F}} = \big\{\lambda[[f(x^s)u(x^s_i)]_{i\in I}]_{x^s \in \text{supp}(f)} \big| f \in \mathcal{F}, \&\ \lambda > 0\big\}.$$

For all $\hat{f}, \hat{g} \in \hat{\mathcal{F}}$, $\hat{f}$ and $\hat{g}$ are said to be *quasi-position preserving* if f and g are position preserving, where $\hat{f} = \lambda[[f(x^s)u(x^s_i)]_{i\in I}]_{x^s\in\text{supp}(f)}$ and $\hat{g} = \mu[[g(y^s)u(y^s_i)]_{i\in I}]_{y^s\in\text{supp}(g)}$. For all $\hat{f} \in \hat{\mathcal{F}}$, define

$$\hat{U}(\hat{f}) = \lambda U(f).$$

STEP 2: $\hat{U}$ is well defined, or for all $\hat{f}, \hat{g} \in \hat{\mathcal{F}}$, if $\hat{f} = \hat{g}$ then $\hat{U}(\hat{f}) = \hat{U}(\hat{g})$.
PROOF OF STEP 2: Choose any $\hat{f}, \hat{g} \in \hat{\mathcal{F}}$ such that $\hat{f} = \hat{g}$ to show $\hat{U}(\hat{f}) = \hat{U}(\hat{g})$. There exist positive numbers λ, μ and $f, g \in \mathcal{F}$ such that $\hat{f} = \lambda[[f(x)u(x_i)]_{i\in I}]_{x\in\text{supp}(f)}$ and $\hat{g} = \mu[[g(y)u(y_i)]_{i\in I}]_{y\in\text{supp}(g)}$. There exist $\text{supp}(f) = \{x^s\}_{s=1}^m$ and $\text{supp}(g) = \{y^s\}_{s=1}^m$ such that $\lambda f(x^s)u(x^s{}_i) = \mu g(y^s)u(y^s_i)$for all $i \in I$ and $s \in \{1, \ldots, m\}$. Define $f' = \dfrac{\lambda}{\lambda+\mu}f + \dfrac{\mu}{\lambda+\mu}f^0$ and $g' = \dfrac{\mu}{\lambda+\mu}g + \dfrac{\lambda}{\lambda+\mu}f^0$. Then

$$\textstyle\sum_{x'\in\mathcal{X}} f'(x')u(x'_1) = \dfrac{\lambda\sum_{x\in\mathcal{X}} f(x)u(x_1)}{\lambda+\mu} = \dfrac{\mu\sum_{y\in\mathcal{X}} g(y)u(y_1)}{\lambda+\mu} = \sum_{y'\in\mathcal{X}} g'(y')u(y'_1).$$

For all $i \in I \setminus \{1\}$,

$$\begin{array}{rcl} \sum_{x'^i(f')} f'(x')[u(x'_i) - u(x'_1)] & = & \dfrac{\lambda}{\lambda+\mu}\sum_{x\in B^i(f)} f(x)[u(x_i) - u(x_1)] \\ & = & \dfrac{\mu}{\lambda+\mu}\sum_{y\in B^i(g)} g(y)[u(y'_i) - u(y_1)] = \sum_{y'^i(g')} g'(y')[u(y'_i) - u(y'_1)], \\ \sum_{x'^i(f')} f'(x')[u(x'_i) - u(x'_1)] & = & \dfrac{\mu}{\lambda+\mu}\sum_{x\in W^i(f)} f(x)[u(x_i) - u(x_1)] \\ & = & \dfrac{\mu}{\lambda+\mu}\sum_{y\in W^i(g)} g(y)[u(y_i) - u(y_1)] = \sum_{y'^i(g')} g'(y')[u(y'_i) - u(y'_1)]. \end{array}$$

Then by Lemma 7, $f' \sim g'$ so that $U(f') = U(g')$. Hence $\dfrac{\lambda}{\lambda+\mu}U(f) + \dfrac{\mu}{\lambda+\mu}U(f^0) = \dfrac{\mu}{\lambda+\mu}U(g) + \dfrac{\lambda}{\lambda+\mu}U(f^0)$, so that $\lambda U(f) = \mu U(g)$. □

For all $\hat{f}, \hat{g} \in \hat{\mathcal{F}}$, define

$$\hat{f} \oplus \hat{g} = (\lambda+\mu)\left[\left[\left[\frac{\lambda f(x^s)}{\lambda+\mu}u(x_i^s)\right]_{i\in I}\right]_{x^s\in \mathrm{supp}(f)}, \left[\left[\frac{\mu g(y^s)}{\lambda+\mu}u(y_i^s)\right]_{i\in I}\right]_{y^s\in\mathrm{supp}(g)}\right],$$

where $\hat{f} = \lambda[[f(x^s)u(x_i^s)]_{i\in I}]_{x^s\in\mathrm{supp}(f)}$ and $\hat{g} = \mu[[g(y^s)u(y_i^s)]_{i\in I}]_{y^s\in\mathrm{supp}(g)}$. I use $\widehat{\sum}_{i=1}^{m}\hat{f}^i$ to denote $\hat{f}^1 \oplus \ldots, \oplus \hat{f}^m$ for all positive integer m.

Step 3: For all $a_1, \ldots, a_K \in \mathbb{R}$ and $\hat{f}^1, \ldots, \hat{f}^K \in \hat{\mathcal{F}}$, if $a_1, \ldots, a_K \geq 0$ and $\hat{f}^1, \ldots, \hat{f}^K$ are pairwise quasi-position preserving then $\hat{U}\Big(\widehat{\sum}_{k=1}^{K} a_k \hat{f}^k\Big) = \sum_{k=1}^{K} a_k \hat{U}(\hat{f}^k)$.

Proof of Step 3:

Claim: Let $K = 2$. If $\hat{f}^1$ and $\hat{f}^2$ are quasi position preserving and $a_1, a_2 \in \mathbb{R}_{++}$ then $\hat{U}(a_1\hat{f}^1 \oplus a_2\hat{f}^2) = a_1\hat{U}(\hat{f}_1) + a_2\hat{U}(\hat{f}^2)$.

Proof of Claim: Choose any $\hat{f}^1, \hat{f}^2 \in \hat{\mathcal{F}}$ and non-negative numbers a_1, a_2. Let $\hat{f}^1, \hat{f}^2$ be quasi position preserving. By the definition of $\hat{\mathcal{F}}$, there exist $f^1, f^2 \in \mathcal{F}$ such that f^1 and f^2 are position preserving and $\hat{f}^1 = \lambda[[f^1(x^s)u(x_i^s)]_{i\in I}]_{x^s\in\mathrm{supp}(f^1)}$ and $\hat{f}^2 = \mu[[f^2(y^s)u(y_i^s)]_{i\in I}]_{y^s\in\mathrm{supp}(f^2)}$ for some positive number λ and μ. Then

$$a_1\hat{f}^1 \oplus a_2\hat{f}^2 = (a_1\lambda + a_2\mu)\left[\left[\left[\frac{a_1\lambda f^1(x^s)}{a_1\lambda+a_2\mu}u(x_i^s)\right]_{i\in I}\right]_{x^s\in\mathrm{supp}(f^1)}, \left[\left[\frac{a_2\mu f^2(y^s)}{a_1\lambda+a_2\mu}u(y_i^s)\right]_{i\in I}\right]_{y^s\in\mathrm{supp}(f^2)}\right].$$

Hence

$$\begin{aligned}
\hat{U}(a_1\hat{f}^1 \oplus a_2\hat{f}^2) &= (a_1\lambda + a_2\mu)U\left(\frac{a_1\lambda}{a_1\lambda+a_2\mu}f^1 + \frac{a_2\mu}{a_1\lambda+a_2\mu}f^2\right) && \left(\because \frac{a_1\lambda}{a_1\lambda+a_2\mu}f^1 + \frac{a_2\mu}{a_1\lambda+a_2\mu}f^2 \in \mathcal{F}\right)\\
&= (a_1\lambda + a_2\mu)\left[\frac{a_1\lambda}{a_1\lambda+a_2\mu}U(f^1) + \frac{a_2\mu}{a_1\lambda+a_2\mu}U(f^2)\right] && (\because f^1 \& f^2 \text{ are position preserving})\\
&= a_1\lambda U(f^1) + a_2\mu U(f^2)\\
&= a_1\hat{U}(\hat{f}^1) + a_2\hat{U}(\hat{f}^2).
\end{aligned}$$

The rest of the proof is induction on K and can be shown by the same way. □

Choose any $f \in \mathcal{F}$. By the end of the proof, I fix this lottery f. Let $I' = I \setminus \{1\}$, $\mathrm{supp}(f) = \{x^1, \ldots, x^m\}$, $I_w = \{i \in I | u(f_1) > u(f_i)\}$, and $I_b = I' \setminus I_w$. For all $i \in I'$, define

$$B_i = \sum_{x\in\mathcal{X}} f(x)\max\{u(x_i) - u(x_1), 0\}, \qquad W_i = \sum_{x\in\mathcal{X}} f(x)\max\{u(x_1) - u(x_i), 0\}.$$

For all positive integer M, define

$$a_1(M) = \frac{\sum_{x\in\mathcal{X}} f(x)u(x_1)}{M}, \quad b_i(M) = \frac{B_i - W_i}{M}, \quad c_i(M) = \frac{2\min\{B_i, W_i\}}{M} \text{ for all } i \in I'.$$

Step 4: There exists a positive integer M^* such that there exist elements $x_1^*, (y_i^*)_{i\in I'}, (z_i^*)_{i\in I'}, ({z^*}'_i)_{i\in I'}$ of X such that $u(x_1^*) = a_1(M^*)$ and $u(y_i^*) = b_i(M^*)$ for all $i \in I'$, $u(z_i^*) = c_i(M^*)$, and $u({z^*}'_i) = -c_i(M^*)$ for all $i \in I'$.

Proof of Step 4: Take enough large $M = M^*$ to hold $1 \geq a_1(M^*) \geq -1$, $1 \geq b_i(M^*) \geq -1$, and $1 \geq c_i(M^*) \geq -1$ for all $i \in I'$. Define

$$\begin{aligned}
x_1^* &= \begin{cases} CE\Big[|a_1(M^*)|\langle x^+\rangle + (1 - |a_1(M^*)|)\langle x^0\rangle\Big] & \text{if } a_1(M^*) \geq 0,\\ CE\Big[|a_1(M^*)|\langle x^-\rangle + (1 - |a_1(M^*)|)\langle x^0\rangle\Big] & \text{if } a_1(M^*) \leq 0,\end{cases}\\
y_i^* &= \begin{cases} CE\Big[|b_i(M^*)|\langle x^+\rangle + (1 - |b_i(M^*)|)\langle x^0\rangle\Big] & \text{if } b_i(M^*) \geq 0,\\ CE\Big[|b_i(M^*)|\langle x^-\rangle + (1 - |b_i(M^*)|)\langle x^0\rangle\Big] & \text{if } b_i(M^*) \leq 0,\end{cases}\\
z_i^* &= CE\Big[|c_i(M^*)|\langle x^+\rangle + (1 - |c_i(M^*)|)\langle x^0\rangle\Big],\\
{z^*}'_i &= CE\Big[|c_i(M^*)|\langle x^-\rangle + (1 - |c_i(M^*)|)\langle x^0\rangle\Big].
\end{aligned}$$

□

For all $i \in I'$ define

$$g^i = \frac{1}{2}\langle z^*_i, (x^0)_{-i}\rangle + \frac{1}{2}\langle {z^*}'_i, (x^0)_{-i}\rangle, \quad \widehat{g^i} = \left[\Big[\frac{u(z^*_i)}{2}, \Big(\frac{u(x^0)}{2}\Big)_{-i}\Big], \Big[\frac{u({z^*}'_i)}{2}, \Big(\frac{u(x^0)}{2}\Big)_{-i}\Big]\right]^{10}.$$

Then $\hat{U}(\widehat{g^i}) = U(g^i)$ for all $i \in I$. For all $i \in I'$ define

$$\widehat{h^i} = \begin{cases} [u(b^i_j)]_{j\in I} & \text{if } u(y^*_i) > 0, \\ [u(x^0), \dots, u(x^0))] & \text{if } u(y^*_i) = 0, \\ [u(w^i_j)]_{j\in I} & \text{if } u(y^*_i) < 0. \end{cases}$$

By definition, $[u(b^i_j)]_{j\in I} = [+1, (0)_{-i}]$, $[u(x^0), \dots, u(x^0))] = (0, \dots, 0)$, and $[u(w^i_j)]_{j\in I} = [-1, (0)_{-i}]$ for all $i \in I'$.

Step 5: $\widehat{h^i}$, and $\widehat{g^j}$ are pairwise quasi position preserving for all $i, j \in I'$.

Proof of Step 5: Note that $i \in I_b$ if $u(y^*_i) > 0$ and $i \in I_w$ if $u(y^*_i) < 0$. By the definition of $\widehat{h^i}$, it is sufficient to show $\{b^i\}_{i\in I_b}, f^0, \{w^i\}_{i\in I_w}$, and $\{g^i\}_{i\in I'}$ are pairwise position preserving.

Substep 5.1: f^0 are position preserving with any other lotteries.

Proof of Substep 5.1: Obvious.

Substep 5.2: For all $i \in I'$, g^i is position preserving with any others.

Proof of Substep 5.2: Since $g^i_j \sim^* \langle x^0\rangle$ for all $i \in I'$ and $j \in I$, Substep 5.2 is established.

Substep 5.3: For all $i, j \in I_b$ such that $i \neq j$, b^i and b^j are position preserving.

Proof of Substep 5.3: $b^i_1 = \langle x^0\rangle = b^j_1$. $b^i_s = \langle x^0\rangle$ or $b^j_s = \langle x^0\rangle$, for all $s \in I \setminus \{1\}$. Hence substep 5.3 is established.

Substep 5.4: For all $i, j \in I_w$ such that $i \neq j$, w^i and w^j are position preserving.

Proof of Substep 5.4: By the same way as Substep 5.4.

Substep 5.5: For all $i \in I_b$ and $j \in I_w$, b^i and w^j are position preserving.

Proof of Substep 5.5: Choose any $b^i \in \{b^i\}_{i\in I_b}$ and $w^j \in \{w^i\}_{i\in I_w}$. Since $I_b = I \setminus I_w$, $i \neq j$. $b^i_1 = \langle x^0\rangle = w^j_1$. $b^i_s = \langle x^0\rangle$ or $w^j_s = \langle x^0\rangle$, for all $s \in I \setminus \{1\}$. Hence substep 5.4 is established □

Define

$$\begin{aligned} \phi &\equiv \frac{1}{2(n-1)+1}\langle\overline{x^*_1}\rangle + \textstyle\sum_{i\neq 1} \frac{1}{2(n-1)+1}\langle y^*_i, (x^0)_{-i}\rangle + \textstyle\sum_{i\neq 1} \frac{1}{2(n-1)+1} g^i, \\ \hat{\phi} &\equiv \frac{1}{2(n-1)+1}[u(x^*_1), \dots, u(x^*_1)] \oplus \widehat{\textstyle\sum}_{i\neq 1} \frac{1}{2(n-1)+1}[u(y^*_i), (u(x^0))_{-i}] \oplus \widehat{\textstyle\sum}_{i\neq 1} \frac{1}{2(n-1)+1}\widehat{g^i}. \end{aligned}$$

Then $\hat{U}(\hat{\phi}) = U(\phi)$. For all $i \in I'$ define

$$\alpha_i = -U(b^i), \quad \beta_i = -U(w^i).$$

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[10] For all $i \in I'$,

$$\widehat{g^i} = \Big(\underbrace{\underbrace{\overbrace{\frac{u(x^0)}{2}, \dots, \frac{u(x^0)}{2}}^{i-1}, \frac{u(z^*_i)}{2}, \overbrace{\frac{u(x^0)}{2}, \dots, \frac{u(x^0)}{2}}^{n-i}}_{n}, \underbrace{\overbrace{\frac{u(x^0)}{2}, \dots, \frac{u(x^0)}{2}}^{i-1}, \frac{u({z^*}'_i)}{2}, \overbrace{\frac{u(x^0)}{2}, \dots, \frac{u(x^0)}{2}}^{n-i}}_{n}}_{2n}\Big).$$

$(\alpha_i)_{i \in I'}$ and $(\beta_i)_{i \in I'}$ are unique because U is unique.
STEP 6:$U(f) = \sum_{x \in \mathcal{X}} f(x)u(x_1) - \sum_{i \in I'} \Big[\alpha_i \max\big\{B_i - W_i, 0\big\} + \beta_i \max\big\{W_i - B_i, 0\big\} - M^* U(g^i)\Big]$.
PROOF OF STEP 6:
SUBSTEP 6.1: $\phi \sim \dfrac{1}{M^*[2(n-1)+1]} f + \left(1 - \dfrac{1}{M^*[2(n-1)+1]}\right) f^0$.

PROOF OF SUBSTEP 6.1: Let $f' = \dfrac{1}{M^*[2(n-1)+1]} f + \left(1 - \dfrac{1}{M^*[2(n-1)+1]}\right) f^0$. By definition,

$$\sum_{x \in \mathcal{X}} f'(x)u(x_1) = \frac{\sum_{x \in \mathcal{X}} f(x)u(x_1)}{M^*[2(n-1)+1]} = \sum_{x \in \mathcal{X}} \phi(x)u(x_1).$$

Choose any $i \in I \setminus \{1\}$. By definition, $u(z_i^*) - u(x^0) \geq 0$ and $u({z^*}'_i) - u(x^0) \leq 0$.
CASE 1: $B_i \geq W_i$. Then $u(y_i^*) - u(x^0) = (B_i - W_i)/M^* \geq 0$. Hence

$$\begin{aligned}
\textstyle\sum_{x \in B^i(\phi)} \phi(x)[u(x_i) - u(x_1)] &= \frac{1}{2(n-1)+1}[u(y_i^*) - u(x^0)] + \frac{1}{2[2(n-1)+1]}[u(z_i^*) - u(x^0)] \\
&= \frac{B_i - W_i}{M^*[2(n-1)+1]} + \frac{2\min\{B_i, W_i\}}{2M^*[2(n-1)+1]} \\
&= \frac{B_i}{M^*[2(n-1)+1]} = \textstyle\sum_{x \in B^i(f')} f'(x)[u(x_i) - u(x_1)]. \\
\textstyle\sum_{x \in W^i(\phi)} \phi(x)[u(x_i) - u(x_1)] &= \frac{1}{2[2(n-1)+1]}[u({z^*}'_i) - u(x^0)] \\
&= \frac{1}{2M^*[2(n-1)+1]}[-2\min\{B_i, W_i\}] \\
&= \frac{-W_i}{M^*[2(n-1)+1]} = \textstyle\sum_{x \in W^i(f')} f'(x)[u(x_i) - u(x_1)].
\end{aligned}$$

Hence by Lemma 7, $\phi \sim f'$.
CASE 2: $W_i \geq B_i$. By the same way as Case 1, it can be shown that $\phi \sim f'$.
SUBSTEP 6.2: $\hat{\phi} = \Big(\dfrac{[u(x_1^*), \ldots, u(x_1^*)]}{2(n-1)+1}, \dfrac{|u(y_2^*)|}{2(n-1)+1}\widehat{h^2}, \ldots, \dfrac{|u(y_n^*)|}{2(n-1)+1}\widehat{h^n}, \dfrac{1}{2(n-1)+1}\widehat{g^2}, \ldots, \dfrac{1}{2(n-1)+1}\widehat{g^n}\Big)$.
PROOF OF SUBSTEP 6.2: By definition.
SUBSTEP 6.3: $U(\phi) = \dfrac{\Big[\sum_{x \in \mathcal{X}} f(x)u(x_1) - \sum_{i \in I'}[\alpha_i \max\{B_i - W_i, 0\} + \beta_i \max\{W_i - B_i, 0\} - M^* U(g^i)]\Big]}{M^*[2(n-1)+1]}$.

PROOF OF SUBSTEP 6.3:

$$U(\phi)$$

$$= \hat{U}(\hat{\phi})$$

$$= \hat{U}\left(\frac{[u(x_1^*), \dots, u(x_1^*)]}{2(n-1)+1}, \frac{|u(y_2^*)|}{2(n-1)+1}\widehat{h^2}, \dots, \frac{|u(y_n^*)|}{2(n-1)+1}\widehat{h^n}, \frac{1}{2(n-1)+1}\widehat{g^2}, \dots, \frac{1}{2(n-1)+1}\widehat{g^n}\right)$$

$(\because$ Substep 6.2)

$$= \frac{1}{2(n-1)+1}\hat{U}(u(x_1^*), \dots, u(x_1^*)) + \textstyle\sum_{i \in I'} \frac{|u(y_i^*)|}{2(n-1)+1}\hat{U}(\widehat{h^i}) + \textstyle\sum_{i \in I'} \frac{1}{2(n-1)+1}\hat{U}(\widehat{g^i})$$

$(\because$ Step 3 and Step 5)

$$= \frac{1}{2(n-1)+1}\Big[U\big(\overline{\langle x_1^* \rangle}\big) + \textstyle\sum_{i \in I_b} |u(y_i^*)|\hat{U}([u(b_j^i)]_{j \in I}) + \textstyle\sum_{i \in I_w} |u(y_i^*)|\hat{U}([u(w_j^i)]_{j \in I}) + \textstyle\sum_{i \in I'} \hat{U}(\widehat{g^i})\Big]$$

$(\because \hat{U}(u(x_1^*), \dots, u(x_1^*)) = U\big(\overline{\langle x_1^* \rangle}\big)$ and if $i \in I_b$ then $\widehat{h^i} = [u(b_j^i)]_{j \in I}$ and if $i \in I_w$ then $\widehat{h^i} = [u(w_j^i)]_{j \in I})$

$$= \frac{1}{2(n-1)+1}\Big[u(x_1^*) + \textstyle\sum_{i \in I_b} |u(y_i^*)|U(b^i) + \textstyle\sum_{i \in I_w} |u(y_i^*)|U(w^i) + \textstyle\sum_{i \in I'} \hat{U}(\widehat{g^i})\Big]$$

$(\because U\big(\overline{\langle x_1^* \rangle}\big) = u(x_1^*), \hat{U}([u(b_j^i)]_{j \in I}) = U(b^i),$ and $\hat{U}([u(w_j^i)]_{j \in I}) = U(w^i)$ for all $i \in I')$

$$= \frac{1}{2(n-1)+1}\Big[u(x_1^*) - \textstyle\sum_{i \in I_b} \alpha_i u(y_i^*) - \textstyle\sum_{i \in I_w} \beta_i[-u(y_i^*)] + \textstyle\sum_{i \in I'} U(g^i)\Big]$$

$(\because U(b^i) = -\alpha_i$, $U(w^i) = -\beta_i$, $\hat{U}(\hat{g^i}) = U(g^i)$ and if $i \in I_b$ then $u(y_i^*) \geq 0$ and if $i \in I_w$ then $u(y_i^*) \leq 0)$

$$= \frac{1}{2(n-1)+1}\Big[u(x_1^*) - \textstyle\sum_{i \in I'} \Big(\alpha_i \max\{u(y_i^*), 0\} + \beta_i \max\{-u(y_i^*), 0\} - U(g^i)\Big)\Big]$$

$$= \frac{1}{M^*[2(n-1)+1]}\Big[\textstyle\sum_{x \in \mathcal{X}} f(x)u(x_1) - \textstyle\sum_{i \in I'} \Big(\alpha_i \max\{B_i - W_i, 0\} + \beta_i \max\{W_i - B_i, 0\} - M^*U(g^i)\Big)\Big].$$

$(\because$ Definitions)

□

For all $i \in I'$, define

$$\gamma_i = -2U\Big(\frac{1}{2}b^i + \frac{1}{2}w^i\Big).$$

$(\gamma_i)_{i \in I'}$ is unique because U is unique.

STEP 7: $U(f) = \sum_{x \in \mathcal{X}} f(x)u(x_1) - \sum_{i \in I'}[\alpha_i \max\{B_i - W_i, 0\} + \beta_i \max\{W_i - B_i, 0\} + \gamma_i \min\{B_i, W_i\}]$.

PROOF OF STEP 7: For all $i \in I'$,

$$\begin{aligned}
-M^*U(g^i) &= -M^*\hat{U}(\hat{g^i}) \\
&= -M^*\hat{U}\left(\left[\frac{u(z_i^*)}{2}, \left[\frac{u(x^0)}{2}\right]_{-i}\right], \left[\frac{u({z^*}'_i)}{2}, \left[\frac{u(x^0)}{2}\right]_{-i}\right]\right) \\
&= -M^*u(z_i^*)\hat{U}\left(\left[\frac{u(x^+)}{2}, \left[\frac{u(x^0)}{2}\right]_{-i}\right], \left[\frac{u(x^-)}{2}, \left[\frac{u(x^0)}{2}\right]_{-i}\right]\right) \qquad (\because u(z_i^*) \geq 0) \\
&= -M^*\frac{2\min\{B_i, W_i\}}{M^*}\hat{U}\left(\frac{[u(b_j^i)]_{j \in I}}{2} \oplus \frac{[u(w_j^i)]_{j \in I}}{2}\right) \\
&= -\min\{B_i, W_i\}2U\Big(\frac{1}{2}b^i + \frac{1}{2}w^i\Big) \\
&= \gamma_i \min\{B_i, W_i\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
U(f) &= M^*[2(n-1)+1]U(\phi) \qquad (\because \text{Substep 6.1}) \\
&= \textstyle\sum_{x \in \mathcal{X}} f(x)u(x_1) - \textstyle\sum_{i \in I'}[\alpha_i \max\{B_i - W_i, 0\} + \beta_i \max\{W_i - B_i, 0\} - M^*U(g^i)] \\
&= \textstyle\sum_{x \in \mathcal{X}} f(x)u(x_1) - \textstyle\sum_{i \in I'}[\alpha_i \max\{B_i - W_i, 0\} + \beta_i \max\{W_i - B_i, 0\} + \gamma_i \min\{B_i, W_i\}]
\end{aligned}$$

□

Step 8: For all $i \in I \setminus \{1\}$,
(i) α_i, β_i, and γ_i are non-negative,
(ii) $\alpha_i + \beta_i \geq \gamma_i$.

Proof of Step 8: Choose any $i \in I \setminus \{1\}$. By IA(i), $f^0 \succ b^i$. Hence $0 = U(f^0) > U(b^i) = -\alpha_i$, so that $\alpha_i > 0$. Similarly, $f^0 \succ w^i$. Hence $0 = U(f^0) > U(w^i) = -\beta_i$, so that $\beta_i > 0$. By IA (ii), $f^0 \succsim (1/2)b^i + (1/2)w^i$. Hence $0 = U(f^0) \geq U((1/2)b^i + (1/2)w^i) = -\gamma_i/2$, so that $\gamma_i \geq 0$.

Moreover by IA(ii), $(1/2)b^i + (1/2)w^i \succsim (1/2)\overline{\langle x_1^i \rangle} + (1/2)\overline{\langle y_1^i \rangle}$ where $\overline{\langle x_1^i \rangle} \sim w^i$ and $\overline{\langle y_1^i \rangle} \sim b^i$ for all $i \in I \setminus \{1\}$. Hence $-\gamma_i \equiv 2U((1/2)b^i + (1/2)w^i) \geq 2U((1/2)\overline{\langle x_1^i \rangle} + (1/2)\overline{\langle y_1^i \rangle}) = U(\overline{\langle x_1^i \rangle}) + U(\overline{\langle y_1^i \rangle}) = U(w^i) + U(b^i) = -(\alpha_i + \beta_i)$, so that $\alpha_i + \beta_i \geq \gamma_i$ for all $i \in I \setminus \{1\}$. □

For all $i \in I'$, define

$$1 - \delta_i = \begin{cases} \dfrac{\gamma_i}{\alpha_i + \beta_i} & \text{if } \alpha_i + \beta_i \neq 0, \\ 0 & \text{if } \alpha_i + \beta_i = 0. \end{cases}$$

Step 9:

$$\begin{array}{rcll} U(f) & = & \sum_{x \in \mathcal{X}} f(x)u(x_1) & - \sum_{i \in I'} \delta_i \alpha_i \max\Big\{ \sum_{x \in \mathcal{X}} f(x)u(x_1) - \sum_{x \in \mathcal{X}} f(x)u(x_i), 0 \Big\} \\ & & & - \sum_{i \in I'} \delta_i \beta_i \max\Big\{ \sum_{x \in \mathcal{X}} f(x)u(x_i) - \sum_{x \in \mathcal{X}} f(x)u(x_1), 0 \Big\} \\ & & & - \sum_{i \in I'} (1 - \delta_i)\alpha_i \sum_{x \in \mathcal{X}} f(x) \max\Big\{ u(x_1) - u(x_i), 0 \Big\} \\ & & & - \sum_{i \in I'} (1 - \delta_i)\beta_i \sum_{x \in \mathcal{X}} f(x) \max\Big\{ u(x_i) - u(x_1), 0 \Big\}. \end{array}$$

Proof of Step 9: Choose any $i \in I'$. By Step 8, $1 \geq \delta_i \geq 0$. By Step 8, if $\alpha_i + \beta_i = 0$ then $\alpha_i = 0 = \beta_i$ and $\gamma_i = 0$. Hence, by definition, $\gamma_i = \delta_i(\alpha_i + \beta_i)$. Hence

$$\begin{array}{rcll} U(f) & = & \sum_{x \in \mathcal{X}} f(x)u(x_1) & - \sum_{i \in I'} \Big[\alpha_i \max\big\{ B_i - W_i, 0 \big\} + \beta_i \max\big\{ W_i - B_i, 0 \big\} + \gamma_i \min\big\{ B_i, W_i \big\} \Big] \\ & = & \sum_{x \in \mathcal{X}} f(x)u(x_1) & - \sum_{i \in I'} \delta_i \Big[\alpha_i \max\big\{ B_i - W_i, 0 \big\} + \beta_i \max\big\{ W_i - B_i, 0 \big\} \Big] \\ & & & - \sum_{i \in I'} (1 - \delta_i) \Big[\alpha_i \max\big\{ B_i - W_i, 0 \big\} \Big]. \end{array}$$

Claim:$\alpha_i \max\big\{ B_i - W_i, 0 \big\} + \beta_i \max\big\{ W_i - B_i, 0 \big\} + (\alpha_i + \beta_i) \min\{B_i, W_i\} = \alpha_i B_i + \beta_i W_i$ for all $i \in I'$.

Proof of Claim: Choose any $i \in I'$.

Case 1: $W_i \geq B_i$. Then L.H.S. $= \beta_i(W_i - B_i) + (\alpha_i + \beta_i)B_i = \alpha_i B_i + \beta_i W_i$.

Case 2: $B_i \geq W_i$. Then L.H.S. $= \alpha_i(B_i - W_i) + (\alpha_i + \beta_i)W_i = \alpha_i B_i + \beta_i W_i$.

By the claim, therefore,

$$U(f) = \sum_{x \in \mathcal{X}} f(x)u(x_1) - \sum_{i \in I'} \Big[\delta_i \alpha_i \max\big\{ B_i - W_i, 0 \big\} + \delta_i \beta_i \max\big\{ W_i - B_i, 0 \big\} + (1 - \delta_i)\alpha_i B_i + (1 - \delta_i)\beta_i W_i \Big].$$

By the definition of B_i and W_i, Step 9 is established. □

Step 10: For all $i, j \in I \setminus \{1\}$,
(iv-a) $\alpha_i \geq \beta_i \Leftrightarrow b^i \precsim w^i$,
(iv-b) $\alpha_i \geq \alpha_j \Leftrightarrow b^i \precsim b^j$,
(iv-c) $\beta_i \geq \beta_j \Leftrightarrow w^i \precsim w^j$,
(iv-d) $1 \geq \beta_i \Leftrightarrow f^- \precsim w^i$.

PROOF OF STEP 10: Choose any $i, j \in I \setminus \{1\}$.

$$\begin{aligned}
\alpha_i \geq \beta_i \quad & \Leftrightarrow \quad U(b^i) \leq U(w^i) && (\because \text{Definition of } \alpha, \beta) \\
& \Leftrightarrow \quad b^i \lesssim w^i. && \\
\alpha_i \geq \alpha_j \quad & \Leftrightarrow \quad U(b^i) \leq U(b^j) && (\because \text{Definition of } \alpha) \\
& \Leftrightarrow \quad b^i \lesssim b^j. && \\
\beta_i \geq \beta_j \quad & \Leftrightarrow \quad U(w^i) \leq U(w^j) && (\because \text{Definition of } \beta) \\
& \Leftrightarrow \quad w^i \lesssim w^j. && \\
1 \geq \beta_i \quad & \Leftrightarrow \quad w^i \geq f^- && (\because \text{Definition of } \beta)
\end{aligned}$$

□

■

C Proof of Corollaries

C.1 Proof of Corollary 1

PROOF OF COROLLARY 1: Choose any $i \in I \setminus \{1\}$. By CIO, there exists $\lambda \in [0,1]$ such that $(1/2)b^i + (1/2)w^i \sim \lambda[(1/2)\overline{\langle x_1^i \rangle} + (1/2)\overline{\langle y_1^i \rangle}] + (1-\lambda)f^0$, where $b^i \sim \overline{\langle x_1^i \rangle}$ and $w^i \sim \overline{\langle y_1^i \rangle}$. Then $-\gamma_i = U((1/2)b^i + (1/2)w^i) = \lambda(1/2)U(\overline{\langle x_1^i \rangle}) + \lambda(1/2)U(\overline{\langle y_1^i \rangle}) + (1-\lambda)U(f^0) = -\lambda(\alpha_i + \beta_i)$. Hence $\delta_i = \lambda$. ■

C.2 Proof of Corollary 2

PROOF OF COROLLARY 1: Choose any $i \in I \setminus \{1\}$. Since $\frac{1}{2}b^i + \frac{1}{2}w^i \sim f^0$, $U\left(\frac{1}{2}b^i + \frac{1}{2}w^i\right) = U(f^0)$. Hence $\gamma_i = 0$. Therefore, by definition, $\delta_i = 0$. ■

C.3 Proof of Corollary 3

PROOF OF COROLLARY 2: Choose any $i \in I \setminus \{1\}$. Since $(1/2)b^i + (1/2)w^i \sim (1/2)\overline{\langle x_1^i \rangle} + (1/2)\overline{\langle y_1^i \rangle}$ where $\overline{\langle x_1^i \rangle} \sim w^i$ and $\overline{\langle y_1^i \rangle} \sim b^i$. Hence $-\gamma_i \equiv 2U((1/2)b^i + (1/2)w^i) = 2U((1/2)\overline{\langle x_1^i \rangle} + (1/2)\overline{\langle y_1^i \rangle}) = U(\overline{\langle x_1^i \rangle}) + U(\overline{\langle y_1^i \rangle}) = U(w^i) + U(b^i) = -(\alpha_i + \beta_i)$, so that $\alpha_i + \beta_i = \gamma_i$. Then $\delta_i = 1$. ■

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