Efficiency results in N player games with imperfect private monitoring*

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Abstract

This paper studies N player infinitely repeated games with imperfect private monitoring where the discount factor is close to unity. Ely and Valimaki [4] (written as EV below) construct a strategy which makes players indifferent among their actions in each period and show the folk theorem in prisoner's dilemma games with two players when monitoring is almost perfect. However, their analysis of the N-player case is limited to a narrow class of games. This paper demonstrates that efficiency is achievable in more general games with three or more players, including a price-setting oligopoly, by introducing a dynamic ex-post equilibrium which is a generalization of EV's. Furthermore, we show that this efficiency result holds with any degree of accuracy if private signals are independent or correlated only through a macro shock. Finally, we spread the available payoff set by modification of a strategy but it is smaller than a feasible and individually rational payoff set.

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1 Introduction

One of the most interesting characteristics of repeated games is the fact that we can achieve efficiency by long-term relationships even in a game where we cannot attain such an outcome in a stage game. There has been a body of literature demonstrating this feature.

Fudenberg and Maskin [6] show the folk theorem in the perfect monitoring case, where the players can observe the opponents' action directly. The folk theorem is also proved by Fudenberg, Levine and Maskin [5] when the monitoring is not perfect but public. However, the case with private monitoring, where players can observe private

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signals about their opponents' actions, is not fully understood, and it has been the subject of active research in the past few years.

There are mainly two kinds of techniques to address this problem. The first one is the belief-based strategy, by Sekiguchi [17] and Bhaskar and Obara [1], and the second is the strategy using the recursive method, by Piccione [16], Obara [15], Kandori and Obara [9] and EV.

EV considers a Markov strategy which makes a player indifferent among his actions whatever the opponents' history is. We call this property *ex-post optimality*. When the opponents play these strategies, players do not have to compute the belief about the history of the opponents because all of his strategies are always optimal. This property provides a recursive structure and allows us to use dynamic programming techniques. In addition, these strategies are robust to a small perturbation in noise. These characteristics make analysis of games with private and almost perfect monitoring drastically simple. Then, EV focuss on the repeated prisoner's dilemma game with two players and derived the folk theorem when the discount factor is close to one and monitoring is almost perfect.

Subsequently, Matsushima [12], [13] and [14] extended these results to games where monitoring is not almost perfect by applying a review strategy. A review strategy induces players to choose a constant action during each T periods, called *review phase*, and to count the number of periods where a particular event occurs in each phase. The action in the next phase is dependent on whether the counted number in the current phase exceeds a certain threshold or not. When T is large enough, each player can detect the action of the opponent almost perfectly from the law of large numbers. He constructed a review strategy in which players are indifferent among his constant actions in each phase by the same way as EV and showed sequential rationality of this strategy by proving that a constant action is optimal under a certain threshold. Thus, he achieved efficiency in a broad class of two player games without the accuracy of monitoring technology.

However, there still remains the question whether efficiency is achivable in general N player games with private monitoring or not. EV had already shown that the implicit function theorem is a powerful tool when we try to apply recursive methods to N player games since it recovers linearity of Bellman equations with respect to transition probabilities. Nevertheless they analyzed only the specified game where cooperation brings each opponent one unit of gain and defection brings himself one unit of gain. Thus, the validity of recursive methods in N player cases is not clear. Furthermore, as far as we know, there are no papers which focus on N player games where monitoring is far from public or accurate. [1]

This paper addresses this question. First of all, we treat almost perfect monitoring cases. We introduce a dynamic ex-post equilibrium, which is the generalization of the equilibria proposed by EV. Since this equilibrium makes players indifferent among their actions, we are able to use dynamic programming techniques analogically. Our method to solve Bellman equations is basically a guess-and-verify approach, in which we regard transition probabilities as fuctions having the discount factor as a variable and apply the implicit function theorem. Then, this paper finds the solution which is different from EV and brings efficiency in many games. We apply this result to several familiar games, such as trading goods and price-setting oligopoly.

[1] Mailath and Morris [10] showed a folk theorem with almost public and almost perfect monitoring. Bhaskar and Obara [1] also derived efficiency results in some class of N player games when monitoring is almost perfect on the assumption that the probability of observing a particular signal profile only depends on the number of errors contained in the signal profile.

Next, we extend this efficiency result to games without almost perfectness of monitoring. It is not obvious whether a review strategy works well in N player games in the following sense. The first problem is that how to detect the opponents' action is not clear. How many kinds of particular events shold we take? And, how should we take thresholds and transition probabilities for weak ex-post optimality? The second problem is that sequential rationality of constant actions is not guaranteed. Indeed, single-peakedness of payoffs brought by non-constant actions is generally broken in N player games whereas Matsushima proved sequential rationality from this property. Roughly speaking, the reason why single-peakedness is broken is as follows. When the number of players is more than two, each player has to pay attention to many kinds of thresholds, namely, player i faces both player j's threshold and player k's threshold. Payoffs brought by non-constant actions make a peak against each threshold and hence the sum of their effects is not single-peaked any longer in N player games. However, we succeed in establishing the review strategy which enables us to overcome such difficulties. Our strategy is suitable for weak ex-post optimality and preserves single-peakedness partially. Here, regarding transition probabilities as fuctions plays a second role in this paper; it is used not only in showing the existence of a review strategy but also in proving sequential rationality since inequalities which guarantees that a review strategy is sequentially rational are not linear. Then, this paper demonstrates that the efficiency results hold with any degree of accuracy if players' private signals are independent or correlated only through a macro shock and each set of private signals is large enough. From this result, we can show that cartel collusion is sustained in a general price-setting oligopoly without accuracy of monitoring.

Finally, we spread the available payoff set. Although the above results generate only symmetric outcomes, we are able to attain asymmetric outcomes by some modifications to strategies. This modified strategy supports a large set of payoffs but it is smaller than a feasible and individually rational payoff set.

The rest of this paper is organized as follows. In section 2, our model is described. Then, we discuss the property of dynamic ex-post equilibria and derive efficiency results when monitoring is accurate in section 3. Section 4 introduces a review strategy and demonstrates that we are able to get efficiency results even when monitoring is far from perfect. Section 5 illustrates the motion on equilibrium path and attains asymmetric payoffs. Finally, section 6 concludes the paper. [2]

2 The Model

The stage game is an N player game $G = \{I, (A_i, \Omega_i, g_i)_{i \in I}, P\}$. $I = \{1, 2, \cdots, N\}$ is the set of players. A_i is the pure action set for player i and $a \in A = \times_{i=1}^{N} A_i$ is the action profile. Suppose that A_i is a finite set and $\{C_i, D_i\} \subseteq A_i$. [3] Ω_i is the finite set of *private* signals for player i and let $\Omega = \times_{i \in I} \Omega_i$. $P(\cdot|a) : \Omega \to [0,1]$ is the conditional probability function of signals. $P(\omega|a)$ stands for the probability that each player i gets ω_i when the current action profile is a where $\omega \in \Omega$. We assume *full support* in that $P(\omega|a) > 0$ for all $a \in A$ and $\omega \in \Omega$. $g_i : A_i \times \Omega_i \to \Re$ is the profit function for player i. $g_i(a_i, \omega_i)$ means the gain when a_i is played and ω_i is realized. Then, define $\pi_i(a) = E[g_i(a_i, \omega_i)|a]$. Note that $\pi_i : A \to \Re$ means the expected payoff for player i.

[2] In addition, we derive efficiency results without symmetricity in Appendix A.

[3] All results in section 3 and Lemma 2 in section 4 still hold if A_i is not a finite set but $\sup_{a \in A} \pi_i(a) < \infty$. When A_i is not countable, μ is not a vector but a function. In this case, μ's convergence is represented by uniform convergence.

Let $\pi_i(C) = 1$ and $\pi_i(D) = 0$.

We consider the infinitely repeated version of G, where the discount factor is $\delta \in (0,1)$. When a_i^t and ω_i^t are the performed action and the observed signals in period t, a history until period T for player i is $h_i^T = (a_i^t, \omega_i^t)_{t=1}^T$. Let H_i^T represent the set of all h_i^T and let $H_i^0 = \phi$. A strategy for player i is a mapping $s_i : \bigcup_{T=0}^{\infty} H_i^T \to \triangle A_i$, where $\triangle A_i$ represents the set of mixed actions for player i. Let S_i be the set of strategies for i and $S = \times_{i=1}^N S_i$. Given $s \in S$, player i's expected average payoff is denoted by $w_i(s)$. Finally, let $A'_i = \{C_i, D_i\}$ and $A^k_{-I'} = \left\{a_{-I'} \in A'_{-I'} \,\middle|\, \sharp\{j | a_j = C_j\} = k\right\}$ for all $0 \le k \le |I| - |I'|$ where $I' \subset I$.

3 Dynamic Ex-post Equilibria

3.1 Efficiency Result

This section treats an almost perfect monitoring case. Suppose that $\Omega_i = A_{-i}$. Then, we can write $\omega_i = (\omega_{i,1}, \ldots, \omega_{i,i-1}, \omega_{i,i+1}, \ldots, \omega_{i,N})$ where $\omega_{i,j}$ is the information on player j's action which player i acquires. Let $\mu = \left(\left(P(\omega|a)\right)_{\omega \neq \times_{i \in I} a_{-i}}\right)_{a \in A}$. Note that μ is the vector of which elements are the probabilities of acquiring the wrong information. We say that monitoring is almost perfect if $\mu \to 0$.

Consider the Markov strategy as follows. In the first period, player i plays C_i or D_i. In each period $t > 1$, player i's mixed action depends only on his own action and the signals in $t-1$. He plays C_i and D_i with probability $1 - \alpha_i(\omega_i)$ and $\alpha_i(\omega_i)$ respectively when his action is C_i and his observed signal is ω_i. He also plays C_i and D_i with probability $\beta_i(\omega_i)$ and $1 - \beta_i(\omega_i)$ when the action is D_i and the signal is ω_i. In the other history, he plays C_i. Let $\gamma_i(\tilde{a}_i, a)$ represent the probability that player i chooses $\tilde{a}_i$ in the next period when his current action is a_i and the signal is a_{-i}. Then, we can write

$$\gamma_i(\tilde{a}_i, a) = \begin{cases} 1 - \alpha_i(a_{-i}) & if \quad (\tilde{a}_i, a_i) = (C_i, C_i) \\ \alpha_i(a_{-i}) & if \quad (\tilde{a}_i, a_i) = (D_i, C_i) \\ \beta_i(a_{-i}) & if \quad (\tilde{a}_i, a_i) = (C_i, D_i) \\ 1 - \beta_i(a_{-i}) & if \quad (\tilde{a}_i, a_i) = (D_i, D_i) \end{cases}.$$

Let Σ_i^* be the set of the above Markov strategies for player i.

Given $\sigma \in \Sigma^*$, let $\sigma_i(a_i)$ represent the strategy which urges player i to play a_i in the first period and then to follow the transition rule of σ_i. Then, a dynamic ex-post equilibrium is defined as follows.

Definition 1 : $\sigma \in \Sigma^*$ is *a dynamic ex-post equilibrium* if

$$\begin{aligned} & {}^{\forall} i \in I \quad {}^{\forall} a_{-i} \in A'_{-i} \quad {}^{\forall} a_i \in A_i \\ & w_i(\sigma_i(C_i), (\sigma_j(a_j))_{j \neq i}) = w_i(\sigma_i(D_i), (\sigma_j(a_j))_{j \neq i}) \ge w_i(\sigma_i(a_i), (\sigma_j(a_j))_{j \neq i}) \end{aligned}$$

Dynamic ex-post equilibria σ are sequential equilibria because a strategy for player i which induces him to choose his action between C_i and D_i in every period is optimal for him when the opponents play σ_{-i}. $\sigma(a)$ is also a sequential equilibrium if $a \in A'$ and σ is a dynamic ex-post equilibrium. We say a dynamic ex-post equilibrium σ sustains $\boldsymbol{V} = \left(\left(V_i(a_{-i})\right)_{a_{-i} \in A'_{-i}}\right)_{i \in I}$ where $V_i(a_{-i})$ denotes the expected average payoff for player i when $\sigma(a)$ is played. Note that dynamic ex-post equilibria in two player games correspond to the equilibria analyzed by EV.

For example, let's look at the two player game, which is treated by EV. Suppose monitoring is perfect, meaning $\mu = 0$. We assume also that $A_i = A'_i$ for each $i \in I$. Since transversality conditions are satisfied for $\delta < 1$, there exists the dynamic ex-post equilibrium σ which sustains $\boldsymbol{V}$ if there are the probabilities α and β which satisfy the following Bellman equations for each $i \in I$ and $j \neq i$;

$$\begin{aligned}
V_i(C_j) &= (1-\delta)\pi_i(C) + \delta\{(1-\alpha_j(C_i))V_i(C_j) + \alpha_j(C_i)V_i(D_j)\}, \\
V_i(C_j) &= (1-\delta)\pi_i(D_iC_j) + \delta\{(1-\alpha_j(D_i))V_i(C_j) + \alpha_j(D_i)V_i(D_j)\}, \\
V_i(D_j) &= (1-\delta)\pi_i(C_iD_j) + \delta\{\beta_j(C_i)V_i(C_j) + (1-\beta_j(C_i))V_i(D_j)\}
\end{aligned}$$

and

$$V_i(D_j) = (1-\delta)\pi_i(D) + \delta\{\beta_j(D_i)V_i(C_j) + (1-\beta_j(D_i))V_i(D_j)\}.$$

The reason of why we can solve the problem of the existence of a dynamic ex-post equilibrium affirmatively is that the number of the equations which must be satisfied is equal to that of the variables, α and β.

Whereas we can investigate two player games easily since equations are linear with respect to variables, there is difficulty in analaysis of the N player game; the equations are not linear any longer. However, EV showed that we can recover linearity by the implicit function theorem. Indeed, Lemma 1, which is the generalization of the results acquired by EV, implies that it suffices to analyze the following linear equations:

$$\sum_{j\neq i} \kappa_j(a)\lambda_j(a)\{v_i(a_{-(i,j)},C_j) - v_i(a_{-(i,j)},D_j)\} = \pi_i(a) - v_i(a_{-i}) \tag{1}$$

for each $i \in I$ and $a \in A'$ where

$$\kappa_j(a) = \begin{cases} -1 & \text{if} \quad a_j = C_j \\ 1 & \text{if} \quad a_j = D_j \end{cases} .$$

Lemma 1 : *Suppose that there exists a vector* $(v_i(a_{-i}))_{i\in I, a_{-i}\in A'_{-i}}$ *satisfying the following properties.*

1. *If* $A_i \neq A'_i$, $\sup_{a_i^* \in A_i \setminus A'_i} \pi_i(a_i^*, D_{-i}) < v_i(D_{-i}) < v_i(a_{-i})$ *for all* $i \in I$ *and* $a_{-i} \neq D_{-i}$.
2. *(1) has a unique solution* $\boldsymbol{\lambda} = (\lambda_i(a))_{i\in I, a\in A'} \ll 0$.

Then,

$$\begin{aligned}
&{}^{\exists}\bar{\delta} < 1 \quad {}^{\forall}\delta \in (\bar{\delta}, 1) \quad {}^{\exists}\bar{\mu} \gg 0 \quad {}^{\forall}\mu (0 \ll \mu \ll \bar{\mu}) \\
&(\textit{there exists the dynamic ex-post equilibrium which sustains } \boldsymbol{V})
\end{aligned}$$

where $V_i(a_{-i}) = v_i(a_{-i})$ *for all* $i \in I$ *and* $a_{-i} \in A'_{-i}$. [4]

proof : See Appendix B.

[4] By letting $\alpha_i(a_{-i})$ be functions which have δ and μ as variables for all $i \in I$ and $a_{-i} \in A_{-i} \setminus A'_{-i}$, we can easily show that the another version of this lemma, that is,

$$\begin{aligned}
&{}^{\exists}\bar{\mu} \gg 0 \quad {}^{\forall}\mu (0 \ll \mu \ll \bar{\mu}) \quad {}^{\exists}\bar{\delta} < 1 \quad {}^{\forall}\delta \in (\bar{\delta}, 1) \\
&(\textit{there exists the dynamic ex-post equilibrium } \sigma \textit{ which sustains } \quad).
\end{aligned}$$

Generally, the set of payoff vectors which are available from Lemma 1 is not a Cartesian product, whereas EV showed the set of payoff vectors which are available by dynamic ex-post equilibria is a Cartesian product in two player prisoner's dilemma. This distinction is from the fact that a dynamic ex-post equilibrium is *interchangeable* only in two player games in the sense that (σ_i, σ'_{-i}) is also a dynamic ex-post equilibrium if σ and σ' are dynamic ex-post equilibria.

It is significantly hard to judge whether efficiency can be attained or not from Lemma 1 although it is the most general result about dynamic ex-post equilibria. Therefore, we'll investigate the sufficient conditions for efficiency results below.

Definition 2 : *g can be achieved approximately with almost perfect monitoring* if for all $\varepsilon > 0$, there exists g' such that $\|g - g'\| < \varepsilon$ and

$$\begin{aligned}&{}^{\exists}\bar{\delta} < 1 \quad {}^{\forall}\delta \in (\bar{\delta}, 1) \quad {}^{\exists}\bar{\mu} \gg 0 \quad {}^{\forall}\mu (0 \ll \mu \ll \bar{\mu}) \\ &(\textit{there exist the equilibria which achieve } g').\end{aligned}$$

Condition 1 : For all $i \in I$, $j \neq i$ and $a \in A'$, $\pi_i(C_i, C_j, a_{-(i,j)}) = \pi_j(C_i, C_j, a_{-(i,j)})$ and $\pi_i(D_i, D_j, a_{-(i,j)}) = \pi_j(D_i, D_j, a_{-(i,j)})$ if $|I| \geq 3$.

Condition 2 : The payoff function satisfies the following properties.

1. For all $i \in I$, $\sup_{a_i \in A_i} \pi_i(a_i, D_{-i}) < \pi_i(C) \leq \pi_i(D_i, C_{-i})$.
2. For all $i \in I$ and $a_{-i} \in A'_{-i} \setminus \{C_{-i}\}$, $\pi_i(C_i, a_{-i}) < \pi_i(C)$.
3. For all $i, j \in I$ and $a \in A'_{-(i,j)} \times \{D_i\} \times \{C_j\}$, $\pi_i(a) > \pi_j(a)$ if $|I| \geq 4$.

Proposition 1 : *Suppose that Conditions* 1 *and* 2 *hold. Then,* $(1, \ldots, 1)$ *can be achieved approximately with almost perfect monitoring.*

proof : Fix real numbers V_k for each $k \in \{0, \ldots, N-1\}$. Consider the dynamic ex-post equilibrium which sustains $\boldsymbol{V}$ satisfying

$$ {}^{\forall}k \in \{0, 1, \ldots, N-1\} \quad {}^{\forall}i \in I \quad {}^{\forall}a_{-i} \in A^k_{-i} \quad V_i(a_{-i}) = V_k. \tag{2}$$

Then, it follows from Condition 1 that the solutions of (1), $\boldsymbol{\lambda}$, satisfies that $\lambda_i(a) = \lambda(C, a)$ when $a_i = C_i$ and $\lambda_i(a) = \lambda(D, a)$ when $a_i = D_i$ for each $a \in A'$. For example, we can re-write a part of (1) and obtain

$$\pi_i(a) - V_{k-1} = (k-1)(V_{k-2} - V_{k-1})\lambda(C, a) + (N-k)(V_k - V_{k-1})\lambda(D, a)$$

and

$$\pi_j(a) - V_k = k(V_{k-1} - V_k)\lambda(C, a) + (N-k-1)(V_{k+1} - V_k)\lambda(D, a)$$

where $k \in \{2, \ldots, N-2\}$, $a_{-j} \in A^k_{-j}$, $a_i = C_i$ and $a_j = D_j$. By solving them, we obtain

$$\lambda(C, a) = \frac{(N-k-1)(V_{k+1} - V_k)(\pi_i(a) - V_{k-1}) - (N-k)(V_k - V_{k-1})(\pi_j(a) - V_k)}{k(N-k)(V_k - V_{k-1})^2 - (N-k-1)(k-1)(V_{k-1} - V_{k-2})(V_{k+1} - V_k)} \tag{3}$$

and

$$\lambda(D, a) = \frac{k(V_k - V_{k-1})(\pi_i(a) - V_{k-1}) - (k-1)(V_{k-1} - V_{k-2})(\pi_j(a) - V_k)}{k(N-k)(V_k - V_{k-1})^2 - (N-k-1)(k-1)(V_{k-1} - V_{k-2})(V_{k+1} - V_k)}. \tag{4}$$

Similarly, we have

$$\lambda(C,C) = \frac{V_{N-1} - \pi_i(C)}{(N-1)(V_{N-1} - V_{N-2})}, \tag{5}$$

$$\lambda(D,D) = \frac{\pi_i(D) - V_0}{(N-1)(V_1 - V_0)}, \tag{6}$$

$$\lambda(C,(D_i,C_{-i})) = \frac{V_{N-1} - \pi_i(D_i,C_{-i})}{(N-1)(V_{N-1} - V_{N-2})}, \tag{7}$$

$$\lambda(D,(C_i,D_{-i})) = \frac{\pi_i(C_i,D_{-i}) - V_0}{(N-1)(V_1 - V_0)}, \tag{8}$$

$$\lambda(D,(D_i,C_{-i})) = \frac{1}{V_{N-1} - V_{N-2}} \left((\pi_j(D_i,C_{-i}) - V_{N-2}) - \frac{(N-2)(V_{N-2} - V_{N-3})(\pi_i(D_i,C_{-i}) - V_{N-1})}{(N-1)(V_{N-1} - V_{N-2})} \right) \tag{9}$$

and

$$\lambda(C,(C_i,D_{-i})) = \frac{1}{V_0 - V_1} \left((\pi_j(C_i,D_{-i}) - V_1) - \frac{(N-2)(V_2 - V_1)(\pi_i(C_i,D_{-i}) - V_0)}{(N-1)(V_1 - V_0)} \right). \tag{10}$$

Take $\{V_k\}_{k=0}^{N-1}$ satisfying $V_{N-1} = 1 - \varepsilon$, $V_{N-1} - V_{N-2} = \varepsilon'$ and $\frac{V_{k+1} - V_k}{V_k - V_{k-1}} = \frac{N-k}{N-k-1}$ for all $k \in \{1, \ldots, N-2\}$ where $\varepsilon > 0$ and $\varepsilon' > 0$. When ε and ε' are small enough, we have $\boldsymbol{\lambda} \ll 0$ from (3),…,(10). □

EV analyzed the following N player game. $A_i = \{C_i, D_i\}$ for $i = 1, \ldots, N$ and $\pi_i(C_i, a_{-i}) + 1 = m_i(a_{-i}) = \pi_i(D_i, a_{-i})$ where $m_i(a_{-i})$ is the number of players different from i that play C in profile a_{-i}. We can get $\pi_i(C_i, a_{-i}) = \frac{m_i(a_{-i}) - 1}{N-2}$ and $\pi_i(D_i, a_{-i}) = \frac{m_i(a_{-i})}{N-2}$ by normalization. Obviously, this game satisfies Conditions 1 and 2. Therefore, we can attain efficiency from Proposition 1. This argument implies our results are the generalization of EV. [5]

Condition 1 requires weak symmetry and Condition 2 is a mild and simple assumption. In the next subsection, we will demonstrate that efficiency is achivable in many games from Proposition 1. Further efficiency results are discussed in Appendix A.

3.2 Examples

Let $a(i,j) = (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_N)$ where $a = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_N)$. Define $\omega(i,j)$ similarly.

Definition 3 : A stage game is *symmetric* if the following properties are satisfied.

1. For each $i, j \in I$, $A_i = A_j$ and $C_i, D_i \in A_i$.
2. For each $i, j \in I$ and $a \in A$, $\pi_i(a) = \pi_j(a(i,j))$.
3. For each $i, j \in I$, $a \in A$ and $\omega \in \Omega$, $\Omega_i = \Omega_j$ and $P(\omega(i,j)|a(i,j)) = P(\omega|a)$.

When the stage game is symmetric, let $\pi_i(C_i, a_{-i}) = c_k$ and $\pi_i(D_i, a_{-i}) = d_k$ for all $i \in I$ and $a_{-i} \in A_{-i}^k$.

[5] They showed that player i can attain payoff v for all $v \in (0,1)$. This result is covered by Proposition 1 and Ellison [2] since the repetition of playing D is a sequential eqilibrium.

3.2.1 Trading Goods

There are N players. In each period, player i produces N goods. Then he exchanges his own products and the others' products one by one and gets all kinds of products. At the end of each period, player i consumes all goods which he has.

Suppose that there are two types in each good; a high quality good and a low quality good. In each period, player i determines an unobservable effort level, which influences on the quality of his products. When he makes an effort, each of his product has high quality and low quality with probability $1-\mu'$ and μ', respectively. When he doesn't make an effort, each of his product has high quality and low quality with probability μ' and $1-\mu'$, respectively. Quality can be observed when the good is consumed. This implies that players can't observe the others' effort level, but he can acquire the private information from quality of goods which he consumed.

Let $A_i = \{C_i, D_i\}$ where C_i means making an effort and D_i means neglecting. We denote by $\Omega_i = \{L, H\}^N$ the set of signals for player i where L means low quality and H means high quality. Let $e > 0$ represent the cost for efforts. Then, we can write $g_i(C_i, \omega_i) = u_i(\omega_i) - e$ and $g_i(D_i, \omega_i) = u_i(\omega_i)$ where $u_i(\omega_i)$ stands for the utility of consuming goods. Suppose that for each $i, j \in I$, $u_i = u_j$ and u_i depends only on the number of high quality goods. We denote by $u(k)$ the utility when he consumes k high quality goods and $N-k$ low quality goods. If for each k, $u(k) < u(k+1)$, $u(0) < u(N) - e < u(N-1)$ and μ' is sufficiently small, Conditions 1 and 2 hold and hence, from Proposition 1, profits in cooperating can be achieved approximately with almost perfect monitoring if these assumptions are satisfied and δ is close to unity.

3.2.2 Price-Setting Oligopoly

There are N sellers and each seller i attaches the price of his own commodities from $A_i = \{0, \ldots, \overline{a}\}$ in each period. They cannnot observe the opponents' price but can recieve their own sales $\omega_i \in \Omega_i$ at the end of the period. Suppose that ω_i is private information. Let $a_{-i}(p)$ be the action profile where each player $j \neq i$ attaches $a_j = p$. When seller i attaches a_i and acquires ω_i, his current gain is $g_i(a_i, \omega_i) = a_i\omega_i - e_i(\omega_i)$ where $e_i(\omega_i)$ stands for the cost of his production. Then, we can define $\pi_i(a) = \sum_{\omega\in\Omega} P(\omega|a) g_i(\omega, a)$.

Assume symmetricity, the single-peakedness, strategic complements and the increasingness of firm's profit functions with respect to their rival's prices in that for each $i, j \in I$, $a \in A$ and $\omega \in \Omega$,

1. $\pi_i(a) = \pi_j(a(i, j))$.
2. $\Omega_i = \Omega_j$ and $P(\omega(i, j)|a(i, j)) = P(\omega|a)$.
3. $\pi_i(a)$ is increasing with respect to a_i when $0 \le a_i < r(a_{-i})$ and decreasing with respect to a_i when $r(a_{-i}) < a_i \le \overline{a}$.
4. $r(a_{-i}) < r(a_{-(i.j)}, a_j + 1)$
5. $\pi_i(a)$ is increasing with respect to a_j.

where $r : A_{-i} \to A_i$ is the best response function. In addition, we assume that π is single-crossing in the sense that for each $a_i, a_i' \in A_i$ and $a_{-i} \in \{\underline{p}, \overline{p}\}^{N-1}$,

$$\pi_i(a_i, a_{-i}) < \pi_i(a_i', a_{-i})$$

if $a_i < a'_i$, $\pi_i(a_i, a_{-i}(\underline{p})) < \pi_i(a'_i, a_{-i}(\underline{p}))$ and $\pi_i(a_i, a_{-i}(\overline{p})) < \pi_i(a'_i, a_{-i}(\overline{p}))$.

Let C_i stand for the cartel price C_i such that $\pi_i(C) \geq \pi_i(a)$ if $a_1 = \cdots = a_N$. Define the defective price D_i as the maximal price p satisfying $\pi_i(C) > \pi_i(r(a_{-i}(p)), a_{-i}(p))$. Then, we can show that for all $i \in I$, $\sup_{a_i \in A_i} \pi_i(a_i, D_{-i}) < \pi_i(C) \leq \pi_i(D_i, C_{-i})$ if $\pi_i(C) < \pi_i(r(D_{-i}), C_{-i})$, which is a natural assumption, in the same way as Matsushima [14]. Moreover, the increasingness of firm's profit and the single-crossing condition guarantees the rest of Condition 2.

Suppose that for each $i \in I$, there exists $\omega_i(k) \in \Omega_i$ which realizes with probability $1 - \mu'$ when $a \in A^k$ is played. Suppose also that for all $i \in I$ and $\omega_i \in \{\omega_i(0), \ldots, \omega_i(N-1)\}$, ω_i occurs with probability less than μ'' when $a \in A \setminus A'$ is played. Note that, from symmetricity, we can easily see that $\alpha_i(a_{-i}) = \alpha_i(a'_{-i})$ and $\beta_i(a_{-i}) = \beta_i(a'_{-i})$ for each $a_{-i}, a'_{-i} \in A^k_{-i}$ when we consider the dynamic ex-post equilibrium which sustains $\boldsymbol{V}$ in Proposition 1. This implies that it is not necessary for player i to distinguish a private signal $a_{-i} \in A^k_{-i}$ from a signal $a'_{-i} \in A^k_{-i}$. Hence, the profits in cartel are sustainable if δ is close to unity and monitoring is almost perfect in the sense that μ' and μ'' are sufficiently small.

3.2.3 Partnership Game with Subjective Assessments

Subjective assessments (or evaluation) are often used in the world. For example, taste of cooking is evaluated subjectively. Private monitoring is one of the suitable tools for this topic and we show one of the examples below.

There are N players who practice a joint work. Player i chooses his action between C_i and D_i where C_i means efforts and D_i means no efforts. Making efforts costs $e > 0$. The work successes and fails with probability $p^*(k)$ and $1 - p^*(k)$ respectively when k players make efforts. When the work successes each player receives $g > 0$. Failure gives players no profits. Suppose that $p^*(k) > p^*(k-1)$ and $gp^*(k) - e < gp^*(k-1)$ for all $k \in \{1, \ldots, N\}$, and $gp^*(0) < gp^*(N) - e$. This implies that D_i dominates C_i and Conditions 1 and 2 are satisfied in this game.

Consider the infinitely repeated version of this game. If players can observe the opponents' action perfectly, we can implement cooperation by the trigger strategy. However, in many cases, there is no objective index which measures the degree of efforts of the opponents directly and players do it according to their own subjective assessments. Can cooperation be attained in this situation?

We establish the formal model. In each period, players acquire the public signal, which is the result of this work, and the private signal, which is the subjective assessment about the opponents' action. Let $\Omega^* = \{S, F\}$ represent the set of public signals and $\Omega_i = A_{-i}$ represent the set of private signals for player i. Public signal $\omega^* \in \Omega^*$ is realized according to p^*. Private signal $\omega \in \Omega = \times_{i \in I} \Omega_i$ follows the conditional probability function $P(\cdot | a, \omega^*) : \Omega \to [0, 1]$. Suppose that the above information structure is common knowledge.

Public perfect equilibria proposed by Fudenberg, Levine and Maskin [5] fails in this game since the set of the public signals is not rich. However, Proposition 1 implies that there exists the dynamic ex-post equilibrium which brings the profits in cooperation if the subjective assessments are almost perfect, meaning that the probabilities that someone acquires wrong private signals are small enough. In this equilibrium, players discard the public signal and use only private information.

4 Review Strategy

This section extends efficiency results in the last section to cases where monitoring is not accurate. Matsushima [12], [13] and [14] showed that a review strategy is a powerful tool in analysis of private monitoring and attained efficiency in some two player games even if monitoring technology is far from accurate. We demonstrate that a review strategy is generalizable and we can achieve efficiency in N player games without accuracy if private signals are independent or correlated only through a macro shock. Note that Ω_i is not A_{-i} any longer here.

In this section, we assume that a signal structure P can be decomposed into two functions denoted by p and f_0 as follows. *A macro shock* θ_0 is randomly drawn according to the conditional probability function $f_0(\cdot|a) : \Theta_0 \to [0,1]$ in each period, where Θ_0 is the finite set of possible macro shocks and a is the action profile which is chosen in this period. A macro shock is unobservable for all players. After a and θ_0 are determined, the private signal ω is randomly drawn according to the conditional probability function $p(\cdot|a,\theta_0) : \Omega \to [0,1]$. Hence, we obtain

$$P(\omega|a) = \sum_{\theta_0 \in \Theta_0} p(\omega|a,\theta_0) f_0(\theta_0|a).$$

Definition 4 : private signals are *correlated only through a macro shock* if P can be decomposed into two functions denoted by p and f_0 and p is conditionally independent for each a and θ_0 in that

$$p(\omega|a,\theta_0) = \prod_{i \in I} p_i(\omega_i|a,\theta_0)$$

for all $\omega \in \Omega$ where $p_i(\omega_i|a,\theta_0)$ is the probability that ω_i is realized when the current action profile is a and the macro shock is θ_0.

In particular, private signals are *independent* if private signals are correlated only through a macro shock and $|\Theta_0| = 1$. We denote by $M(\Omega,\Theta_0,p,f_0)$ the above signal structure and given Ω and Θ_0, let $M^*(\Omega,\Theta_0)$ represent the set of all $M(\Omega,\Theta_0,p,f_0)$.

Those who play a review strategy divide infinite periods for every T periods, called *review phase*, and play constant actions in each phase. The action in the next phase depends only on his own action and the gathered information, called random event, during the current phase. According to the law of large numbers, players acquire almost perfect information about the opponents' actions by fixing T large enough since they play constant actions in each phase. Here, we construct a review strategy which has weak ex-post optimality in the sense that players' action is chosen between C_i and D_i in each phase and they are indifferent between continuing to play C_i T times and to play D_i T times in the initial period of each review phase. Accuracy of monitoring in each phase guarantees the existence of this strategy on analogy of the previous section. Weak ex-post optimality allows that this strategy profile is an equilibrium if deviations in the first phase brings no profits.

Given σ^T, which is a review strategy where each review phase is T periods, let $\sigma_i^T(a_i)$ represent a review strategy for player i which induces her to play a_i in the first review phase and then to follow the transition rule of σ_i^T. Weak ex-post optimality requires that

$$^\forall i \in I \quad ^\forall a_{-i} \in A'_{-i} \quad w_i(\sigma_i^T(C_i),\sigma_{-i}^T(a_{-i})) = w_i(\sigma_i^T(D_i),\sigma_{-i}^T(a_{-i})) = V_i(a_{-i}). \quad (11)$$

We say that the review strategy σ^T generates $\boldsymbol{V}$ if (11) holds. We say σ^T sustains $\boldsymbol{V}$ if σ^T generates $\boldsymbol{V}$ and σ^T is a sequential equilibrium.

We define *a random event* as a function $\psi_i : \Omega_i \to [0,1]$, whose definition is due to Matsushima [14]. In each period, player i chooses the real number $x(\psi_i)$ from $[0,1]$ according to the uniform distribution and count a random event ψ_i if $x(\psi_i) \le \psi_i(\omega_i)$. Then, we can say that a random event ψ_i occurs with probability $\psi_i(\omega_i)$ when player i observes the signal ω_i. Player's action in the next phase relies not only on his current action but also on the number of random events gatherd during the current phase. Let $q(\psi_i|a)$ and $q(\psi_i|a,\omega_{-i})$ represent the probability that ψ_i occurs when action a is played and the one that ψ_i occurs when action a is played and the opponents observe ω_{-i}, respectively.

Let $\{a_{-i}(k)\}_{k=1}^{2^{N-1}}$ and $\{\tilde{a}_{-i}(k)\}_{k=1}^{2^{N-1}}$ are sequences satisfying $a_{-i}(k) \in A'_{-i}$, $\tilde{a}_{-i}(k) \in A'_{-i}$, $a_{-i}(k) \ne a_{-i}(l)$ and $\tilde{a}_{-i}(k) \ne \tilde{a}_{-i}(l)$.

Condition 3 : For each $\{a_{-i}(k)\}_{k=1}^{2^{N-1}}$ and $\{\tilde{a}_{-i}(k)\}_{k=1}^{2^{N-1}}$, there exists random events $\{\psi_i^C(k)\}_{k=1}^{2^{N-1}-1}$, $\{\psi_i^D(k)\}_{k=2}^{2^{N-1}}$, ψ_i^+ and ψ_i^{++} for all $i \in I$ which satisfy the following properties.

1. For all $l \in \{1,\dots,2^{N-1}\}$,
$$q(\psi_i^C(k)|C_i,a_{-i}(l)) = \begin{cases} \overline{q} & if \quad l > k \\ \underline{q} & if \quad l \le k \end{cases},$$
$$q(\psi_i^D(k)|D_i,\tilde{a}_{-i}(l)) = \begin{cases} \overline{q} & if \quad l < k \\ \underline{q} & if \quad l \ge k \end{cases},$$
for all $j \ne i$, $a_{-(i,j)} \in A'_{-(i,j)}$ and $a_j \in A_j \setminus A'_j$,
$$q(\psi_i^C(k)|C_i,a_{-i}) = q(\psi_i^C(k)|C_i,D_j,a_{-i,j}),$$
for all $j \ne i$, $a_{-(i,j)} \in A'_{-(i,j)} \setminus \{D_{-(i,j)}\}$ and $a_j \in A_j \setminus A'_j$,
$$q(\psi_i^D(k)|D_i,a_{-i}) = q(\psi_i^D(k)|D_i,D_j,a_{-i,j}),$$
$$q(\psi_i^D(k)|D_{-j},a_j) = \overline{q},$$
for all $a_{-i} \notin A'_{-i}$ and $a'_{-i} \in A'_{-i}$,
$$q(\psi_i^+|C_i,a'_{-i}) = q(\psi_i^{++}|D_i,a'_{-i}) = \underline{q},$$
and
$$q(\psi_i^+|C_i,a_{-i}) = q(\psi_i^{++}|D_i,a_{-i}) = \overline{q}.$$
where $\overline{q} > \underline{q}$.

2. For every $a_{-i} \in A_{-i}$ and $\omega_{-i} \in \Omega_{-i}$,
$$q(\psi_i^C(k)|C_i,a_{-i},\omega_{-i}) = q(\psi_i^C(k)|C_i,a_{-i}),$$
$$q(\psi_i^D(k)|D_i,a_{-i},\omega_{-i}) = q(\psi_i^D(k)|D_i,a_{-i}),$$
$$q(\psi_i^+|C_i,a_{-i},\omega_{-i}) = q(\psi_i^+|C_i,a_{-i})$$
and
$$q(\psi_i^{++}|D_i,a_{-i},\omega_{-i}) = q(\psi_i^{++}|D_i,a_{-i}).$$

Lemma 2 : *Suppose that there exists $\boldsymbol{v}^*$ satisfying the conditions in Lemma* 1 *and* $v_i^*(C_j, a_{-(i,j)}) > v_i^*(D_j, a_{-(i,j)})$ *for all* $i, j \in I$ *and* $a_{-(i,j)} \in A'_{-(i,j)}$*. Then, if Condition* 3 *holds, there exists a review strategy which sustains* $\boldsymbol{V} = \boldsymbol{v}^*$ *for* δ *close to unity.*

proof : See Appendix C.

We construct a review strategy used in this lemma as follows. Take sequences $\{a_{-i}(k)\}_{k=1}^{2^{N-1}}$ and $\{\tilde{a}_{-i}(k)\}_{k=1}^{2^{N-1}}$ satisfying

$$\lambda_i(C_i, a_{-i}(k)) > \lambda_i(C_i, a_{-i}(k+1))$$

and

$$\lambda_i(D_i, \tilde{a}_{-i}(k)) > \lambda_i(D_i, \tilde{a}_{-i}(k+1))$$

where $\boldsymbol{\lambda}$ is defined by substituting $\boldsymbol{v} = \boldsymbol{V}$ into (1). [6] Take random events according to Condition 3.

Fix a sufficintly large integer $T > 0$ and take an integer Z_T satisfying $T\underline{q} < Z_T < T\overline{q}$. We denote by $X_i(C, k)$ the event that $\psi_i^C(k)$ occurs less than $Z_T + 1$ times during the last T periods when she has played C_i during the same periods. Similarly, $X_i(D, k)$ represents the event that $\psi_i^D(k)$ occurs less than $Z_T + 1$ times during the last T periods when she has played D_i during the same periods. Denote by X_i^+ the event that ψ_i^+ occurs less than $Z_T + 1$ times during the last T periods when she has played C_i during the same periods. Finally, denote by X_i^{++} the event that ψ_i^{++} occurs less than $Z_T + 1$ times during the last T periods when she has played D_i during the same periods.

Consider a review strategy as follows. In the first period, player i chooses either C_i or D_i. In period $nT + 1$, player i chooses C_i and D_i with probability $1 - \alpha_i^+$ and α_i^+ respectively if she has played C_i and no events have occured in the last T periods. When $X_i(C, k)$ occurs, we add $\alpha(k)$ to the probability that she plays C_i in the current period. When X_i^+ occurs, we add $\alpha(2^{N-1})$ to the probability that she plays C_i in the current period. She chooses C_i and D_i with probability zero and one respectively if she has played D_i and no events have occured in the last T periods. When $X_i(D, k)$ occurs, we add $\beta(k)$ to the probability that she plays C_i in the current period. When X_i^{++} occurs, we add $\beta(1)$ to the probability that she plays C_i in the current period. In the other history, she plays C_i. In period t where $t \neq nT + 1$, she plays the action which she played immediately before.

For example, when $X_i(C, 1)$, $\ldots$, $X_i(C, 2^{N-1} - 1)$ and X_i^+ occur during the last T periods, player i chooses C_i and D_i with probability $1 - \alpha_i^+ + \sum_{l=1}^{2^{N-1}} \alpha(l)$ and $\alpha_i^+ - \sum_{l=1}^{2^{N-1}} \alpha(l)$, respectively.

We call this T period interval a review phase. Let Σ_i^T be the set of the above strategies for player i. We use a review strategy $\sigma^T \in \Sigma^T$ satisfying (11). According to the law of large numbers, we are able to regard the judgement as almost perfect monitoring when everyone choose a constant action in each phase. Then, it follows by the same logic as in the almost perfect monitoring case that there exists a review strategy σ^T satisfying (11) when δ is close to unity and T is large enough.

We can complete the proof by showing sequential rationality of σ^T. Note that Condition 3-2 implies that the private signal ω_i has no information about whether the random events for the opponents occur or not. Therefore, it suffices to show that each

[6] Even when $\lambda_i(C_i, a_{-i}) = \lambda_i(D_i, a_{-i})$ for some $i \in I$ and $a_{-i} \in A_{-i}$, we can prove this lemma with a few corrections.

player doesn't want to deviate during the first T periods with history independent action choices. Note that occurance of events of the opponents brings her profits since $V_i(C_j, a_{-(i,j)}) > V_i(D_j, a_{-(i,j)})$. Hence, by fixing $\frac{Z_T}{T}$ close to $\underline{q}$, even deviating one time makes the probabilities of the occurance of these events sufficiently low and reduces her continuation payoff. In this way, we can show that σ^T is sequentially rational for some Z_T.

The important technique in this proof is to regard α and β as differentiable functions which have the discount factor as a variable like section 3 not only in showing the existence of the review strategy but also in confirming sequential rationality since the loss of deviations is not linear with respect to the transition probabilities any longer when there are three or more players. By regarding α and β as functions which satisfy $\alpha = \beta = 0$ with $\delta^T = 1$, player's average payoff defined by Bellman equations are indifferent among all of their strategies when $\delta^T = 1$. Hence, all we have to do in order to say that σ^T is a sequential equilibrium is to show that derivatives of gains of when she deviates are positive.

To prove such a property, we make all thresholds the same number. This preserves single-peakedness of i's payoffs brought by her own deviations in the initial phase partially. Since all thresholds are the same number and all probabilities that a certain random event occurs are either high or low, there are only two kinds of the numbers of deviations during a review phase which maximizes the probabilities that a certain random event occurs just Z_T times. Note that these two numbers are separated enough each other. Hence, it follows that a sum of such probabilities, which determines a continuation payoff, has only two peaks with respect to the number of deviations and these peaks have enough distance. Then, we are able to prove sequential rationality of a review strategy from this property.

In our strategy, events are interpretted as follows. When $X_i(C,k),\ldots, X_i(C, 2^{N-1} - 1)$ and X_i^+ occur in some review phase, she judges $a_{-i}(k)$ is played in that phase. When $X_i(D,2),\ldots, X_i(D,k)$ and X_i^{++} occur, she judges $\tilde{a}_{-i}(k)$ is played in that phase also. This design seems to be strange but plays a important role. For example, consider the random event $\psi_i^C(k)$ which occurs with low probability only when $a_{-i}(k)$ is performed. When $X_i(C,k)$ occurs, player i adds $\alpha(k)$ to the probability that she plays C_i where α is defined for weak ex-post optimality. Here, event $X_i(C,k)$ stands for $a_{-i}(k)$ and this design seems to be natural. However, according to this strategy, player i must choose C_i with probability larger than one when many events occured simultaneously. This implies that such a strategy cannot exist. If we define the probability that player i choose C_i when two or more events occur simultaneously, which is not the sum of $alpha$, in order to avoid non-existence of this strategy, our analysis becomes drastically difficult since we must calculate the probability that both $X_i(C,k)$ and $X_i(C,k+1)$ occur.

We have improved a review strategy compared with Matsushima's strategy; even the constant defection may increase continuation payoffs here. This spreads the class of games where the efficiency result holds. In two player games, Matsushima assumed for efficiency that

$$\max_{a_i \in A_i} \pi_i(a_i, D_j) < \pi_i(C) < \pi_i(D_i C_i)$$

and

$$\pi_i(C_i, D_i) < \pi_i(D)$$

but Lemma 2 requires only

$$\max_{a_i \in A_i} \pi_i(a_i, D_j) < \pi_i(C) \leq \pi_i(D_i C_i)$$

when we want to approximate $(\pi_1(C), \pi_2(C))$. [7]

We'll examine the sufficient condition for Condition 3 below. Suppose that for every $a_i \in A'_i$, $\{p_i(\cdot|a,\theta_0)|(a_{-i},\theta_0) \in A_{-i} \times \Theta_0\}$ is *linearly independent* in the sense that there exists no function $e' : A_{-i} \times \Theta_0 \to \Re$ such that

$$(e'(a_{-i},\theta_0))_{(a_{-i},\theta_0)\in A_{-i}\times\Theta_0} \neq 0$$

and

$$\sum_{(a_{-i},\theta_0)\in A_{-i}\times\Theta_0} e'(a_{-i},\theta_0)p_i(\cdot|a,\theta_0) = 0.$$

Then, there exist $\psi_i^C(k)$, $\psi_i^D(k)$, ψ_i^+ and ψ_i^{++} such that Condition 3-1 holds and for every $i \in I$, $a_{-i} \in A_{-i}$ and $\theta_0 \in \Theta_0$,

$$\begin{aligned}
q(\psi_i^C(k)|C_i,a_{-i},\theta_0) &= q(\psi_i^C(k)|C_i,a_{-i}),\\
q(\psi_i^D(k)|D_i,a_{-i},\theta_0) &= q(\psi_i^D(k)|D_i,a_{-i}),\\
q(\psi_i^+|C_i,a_{-i},\theta_0) &= q(\psi_i^+|C_i,a_{-i})
\end{aligned}$$

and

$$q(\psi_i^{++}|D_i,a_{-i},\theta_0) = q(\psi_i^{++}|D_i,a_{-i})$$

where $q(\psi_i|a,\theta_0)$ represents the probability that ψ_i occurs when the current action profile is a and a macro shock is θ_0. By the same method as Lemma 3 in Matsushima [14], we can say that Condition 3-2 also holds.

Linearly independence holds almost everywhere in $M^*(\Omega,\Theta_0)$ if

$$|\Omega_i| \geq |A_{-i}| \times |\Theta_0| \tag{12}$$

for all $i \in I$. Then, we obtain the following theorem.

Theorem : *Suppose that there exists $\boldsymbol{v}^*$ satisfying the conditions in Lemma* 1 *and $v_i^*(C_j,a_{-(i,j)}) > v_i^*(D_j,a_{-(i,j)})$ for all $i,j \in I$ and $a_{-(i,j)} \in A'_{-(i,j)}$. Then, if private signals are correlated only through a macro shock and inequalities (12) hold, there exists a review strategy which sustains $\boldsymbol{V} = \boldsymbol{v}^*$ for δ close to unity almost everywhere in $M^*(\Omega,\Theta_0)$.*

Corollary 1 : *If private signals are correlated only through a macro shock and Conditions* 1 *and* 2 *and inequalities (12) hold, we can attain* $(1,\ldots,1)$ *approximately for δ close to unity almost everywhere in $M^*(\Omega,\Theta_0)$.*

Remark 1 : We are able to relax inequalities (12). Let $\tilde{A}_{-i} = A_{-i} \setminus \hat{A}_{-i}$ where $\hat{A}_{-i}$ is the maximal set such that for all $a_{-i} \in \hat{A}_{-i}$, there exist $j,k \in I$ satisfying $a_j \notin A'_j$ and $a_k \notin A'_k$. We require only linearly independence of $\{p_i(\cdot|a,\theta_0)|(a_{-i},\theta_0) \in \tilde{A}_{-i} \times \Theta_0\}$ in this equilibrium. Hence, we can replace (12) by $|\Omega_i| \geq |\tilde{A}_{-i}| \times |\Theta_0|$.

Remark 2 : When the stage game is symmetric, $\{p_i(\cdot|a,\theta_0)|(a_{-i},\theta_0) \in \tilde{A}_{-i} \times \Theta_0\}$ is not linearly independent. However, from symmetricity, player i doesn't need to distinguish between $a_{-i} \in A^k_{-i}$ or $a'_{-i} \in A^k_{-i}$. Therefore, efficiency results hold in symmetric games if $|\Omega_i|$ is large enough.

[7] Use satisfying $V_i(C_j) = \pi_i(C) - \varepsilon$ and $V_i(D_j) = \pi_i(C) - 2\varepsilon$ where ε is small enough.

Corollary 1 enables us to attain efficiency in the price-setting oligopoly which appeared in the previous subsection even when monitoring is not almost perfect if $|\Omega_i|$ is large enough and private signals are correlated only through a macro shock. [8] In the trading goods game, we can attain efficiency also if Conditions 1 and 2 hold and private signals are independent, meaning $|\Theta_0| = 1$. This independence holds in the example of the previous subsection since the quality of goods are independently determined. Furthermore, in the partnership game, we can make public signals have no information on the random events of the opponents if the subjective assessment is conditionally independent and the set of the private signals is large enough. Then, cooperation is achievable there.

5 Motion on Equilibria and Asymmetric Outcomes

In the previous sections, we have shown the existence of equilibria which sustains near efficiency. We have not yet illustrated such equilibria, though, explicitly. It is difficult to get the transition probabilities, α and β, in three or more player games even if monitoring is perfect since it entails calculation of Bellman equations without the implicit function theorem but these equations are not linear with respect to α and β. However, the following proposition says something about this feature.

Proposition 2 : *Suppose that Conditions* 1 *and* 2 *hold and monitoring is perfect. Let* $\boldsymbol{V}(\varepsilon,\varepsilon')$ *be the balue function appearing in Proposition* 1*. Then, for all* $\eta > 0$*, there exists* $(\bar{\varepsilon},\bar{\varepsilon}') \gg 0$ *such that for all* $\varepsilon < \bar{\varepsilon}$ *there exists* σ *such that* σ *sustains* $\boldsymbol{V}(\varepsilon,\bar{\varepsilon}')$ *for sufficiently large* δ *and* $\lim_{\delta\to 1} \frac{\delta}{1-\delta}\alpha_i(C_{-i}) < \eta$*.*

proof : The existence of σ which sustains $\boldsymbol{V}(\varepsilon,\bar{\varepsilon}')$ is clear. From (5), we have

$$-\left.\frac{\partial \alpha_i(C_{-i})}{\partial \delta}\right|_{\delta=1} = \frac{\varepsilon}{(N-1)\bar{\varepsilon}'} < \eta$$

for sufficiently small ε. Note that $\alpha_i(C_{-i})$ is the function of which value is zero for $\delta = 1$. Then, we obtain

$$-\left.\frac{\partial \alpha_i(C_{-i})}{\partial \delta}\right|_{\delta=1} = \lim_{\delta\to 1}\frac{\alpha_i(C_{-i})}{1-\delta}.$$

Since $\frac{\delta\alpha_i(C_{-i})}{1-\delta} < \frac{\alpha_i(C_{-i})}{1-\delta}$ for $\delta < 1$, we have finished the proof. □

The implication of Proposition 2 is that the probability that player i plays D_i when the current action profile is C is too small to affect on the average payoff if the dynamic equilibrium sustains the almost efficient outcome. Note that α is continuous with respect to noises. Therefore, this property holds for sufficiently large δ with any degree of accuracy since long review phases eliminate the noise.

We can expand the set of available payoff vectors by using this property. For our ease, consider symmetric games where each player chooses his action between cooperation and defection. The payoff function is *monotonic with respect to C* if $\pi_i(a_i,a_{-i}) < \pi_i(a_i,a'_{-i})$ for all $a_i \in A_i$, $k \in \{0,\dots,N-2\}$, $a_{-i} \in A^k_{-i}$ and $a'_{-i} \in A^{k+1}_{-i}$. Let $\pi(a) = (\pi_i(a))_{i\in I}$.

[8] Matsushima [14] display the sufficient conditions for linearly independence in price-setting duopoly.

Proposition 3 : *Suppose that the stage game is symmetric, $A_i = \{C_i, D_i\}$, the payoff function is monotonic with respect to C and D_i dominates C_i strictly. Then, $\pi(a^*) \gg 0$ can be achieved approximately with almost perfect monitoring. Furthermore, if private signals are correlated only through a macro shock and $|\Omega_i|$ is large enough, we can attain $\pi(a^*) \gg 0$ approximately for δ close to unity almost everywhere in $M^*(\Omega, \Theta_0)$.*

proof : We treat only the case without accuracy. Construct the following equilibrium. Let $\bar{I} = \{i \in I | a_i^* = C_i\}$ and $K = |\bar{I}|$. Player $i \notin \bar{I}$ chooses D_i infinitely. Player $i \in \bar{I}$ plays a review strategy which satisfies weak ex-post optimality among $\bar{I}$ and sustains $\bar{\boldsymbol{V}} = \left(\left(\bar{V}_i(a) \right)_{a \in (A_j)'_{j \in \bar{I} \setminus \{i\}}} \right)_{i \in \bar{I}}$ satisfying (2), $V_{K-1} = c_{K-1} - \varepsilon$, $V_{K-1} - V_{K-2} = \varepsilon'$ and $\frac{V_{k+1} - V_k}{V_k - V_{k-1}} = \frac{K-k}{K-k-1}$. We can prove the existence of such an equilibrium in the same way as Proposition 1 and Lemma 2.

Then, it follows from Proposition 2 that the average payoff of player $i \notin \bar{I}$ converges to d_K as ε goes to zero. □

According to Ellison [2], we can attain the convex hull of outcomes supported by Proposition 3.

6 Conclusion

This paper has established the efficiency results in N player repeated games with private monitoring. First, we have defined a dynamic ex-post equilibrium in almost perfect monitoring games. Since a dynamic ex-post equilibrium makes players always indifferent among their actions, we don't need to compute their belief about opponents' histories. This makes our analysis simple and we have been able to derive the efficiency results in many games. Then, we have extend these results by applying a review strategy and achieved efficiency without accuracy of monitoring technology. As applications of our results, we have analyzed several familiar games and shown that efficiency can be achieved generally even when monitoring is imperfect and private. Finally, we have expanded the available payoff set.

However, we have not shown the folk theorem. Our strategy can attain only a part of a feasible and individually rational payoff set. Furthermore, these results are available only if the payoff structure is like a prisoner's dilemma. When the game is far from prisoner's dilemma and has no suitable solutions of (1), there seems to be severe difficulty in analysis. [9] Although we might be able to achieve efficiency with ex-post optimal Markov strategies which have three or more states or non-Markovian strategies, it is beyond the purpose of this paper. [10]

Repeated games with private monitoring still contain many unsolved problems, and progress of future research is expected.

[9] Yamamoto [18] showed that a public randomization device gives us extended efficiency results.

[10] Recently, Ely, Horner and Olszewski [3] introduced the concept of belief-free equilibria and provided a characterization of equilibrium payoffs in general two player games by allowing players access to a public randomization device. Dynamic ex-post equilibria are belief-free in their sense.

Appendix A : Extension of Efficiency Results

Proposition 1 is useful but it requires weak symmetricity, corresponding Condition 1. Here, we consider efficiency results when symmetricity is broken. Then, we extend it by applying a review strategy. Let $\bar{\pi}(a) = \frac{1}{N}\sum_{i\in I}\pi_i(a)$.

Condition 4 : The payoff function satisfies the following properties.

1. For all $i \in I$, $\sup_{a_i \in A_i} \pi_i(a_i, D_{-i}) < \pi_i(C)$.
2. For all $i \in I$ and $a_{-i} \in A'_{-i} \setminus \{C_{-i}\}$, $\pi_i(C_i, a_{-i}) < \pi_i(C)$ and $\bar{\pi}(C_i, a_{-i}) < \pi_i(C)$.
3. For all $i, j \in I$ and $a_{-(i,j)} \in A'_{-(i,j)}$,
$$\pi_i(D_i, C_j, a_{-(i,j)}) > \bar{\pi}(D_i, C_j, a_{-(i,j)}) > \pi_j(D_i, C_j, a_{-(i,j)}).$$
4. For all $i, j \in I$,
$$\pi_i(D_i, C_{-i}) - \pi_i(C) \geq 2\left((N-1)\pi_j(D_i, C_{-i}) - \sum_{l\neq i}\pi_l(D_i, C_{-i})\right).$$

Proposition 4 : *Let the setting of information structure be the same as in section 3. Then,* $(1,\ldots,1)$ *can be achieved approximately with almost perfect monitoring if Condition* 4 *holds.*

proof : Consider the dynamic ex-post equilibrium which sustains $\boldsymbol{V}$ appearing in Proposition 1 and use Lemma 1. □

Corollary 2 : *Suppose that private signals are correlated only through a macro shock and Condition* 4 *holds. Then, if inequalities (12) hold, we can attain* $(1,\ldots,1)$ *approximately for* δ *close to unity almost everywhere in* $M^*(\Omega, \Theta_0)$.

Appendix B : The Proof of Lemma 1

Given $\sigma \in \Sigma^*$, $\tilde{a}_{-i} \in A'_{-i}$, $a_i \in A_i$ and $a_{-i} \in A'_{-i}$, let

$$\zeta(\tilde{a}_{-i}, a, \sigma, \mu) = \sum_{\omega_{-i} \in \Omega_{-i}} P(\omega_{-i}|a) \prod_{j \neq i} \gamma_j(\tilde{a}_j, (a_j, \omega_j)).$$

Note that $\zeta(\tilde{a}_{-i}, a, \sigma, \mu)$ means the probability that $\tilde{a}_{-i}$ is played in the next period when the opponents play σ_{-i} and the current action profile is a. Let $E_i[\boldsymbol{V}|a, \sigma, \mu]$ represent the expected value when $V_i(\tilde{a}_{-i})$ is obtained with probability $\zeta(\tilde{a}_{-i}, a, \sigma, \mu)$.

We can say σ is the dynamic ex-post equilibrium which sustains $\boldsymbol{V}$ if

$$W_i(a, \boldsymbol{V}, \sigma, \mu) \leq V_i(a_{-i}) \tag{13}$$

for all $i \in I$ and $a \in A_i \times A'_{-i}$ with equalities for all $i \in I$ and $a \in A'$, where

$$W_i(a, \boldsymbol{V}, \sigma, \mu) = (1-\delta)\pi_i(a) + \delta E_i[\boldsymbol{V}|a, \sigma, \mu].$$

When $A \neq A'$, fix $\alpha_i(a_{-i}) = 1$ and $\beta_i(a_{-i}) = 0$ for all $i \in I$ and $a_{-i} \in A_{-i} \setminus A'_{-i}$. Then, (13) hold with strict inequalities for all $a \in A \setminus A'$ when δ is close to one and μ is small enough. Hence, it suffices to show that there are probabilities $((\alpha_i(a_{-i}))_{a_{-i} \in A'_{-i}})_{i \in I}$ and $((\beta_i(a_{-i}))_{a_{-i} \in A'_{-i}})_{i \in I}$ which satisfy (13) with equalities for all $i \in I$ and $a \in A'$.

When $((\alpha_i(a_{-i}), \beta_i(a_{-i}))_{a_{-i} \in A'_{-i}})_{i \in I} = 0$, we have $E_i[\boldsymbol{V}|a, \sigma, 0] = V_i(a_{-i})$ for all $i \in I$ and $a \in A'$. Therefore, every $\boldsymbol{V}$ satisfies (13) with equalities for all $i \in I$ and $a \in A'$ when $\delta = 1$, $\mu = 0$ and $\alpha_i(a_{-i}) = \beta_i(a_{-i}) = 0$ for all $a_{-i} \in A'_{-i}$. According to the implicit function theorem, these equations hold for δ close to unity by regarding $((\alpha_i(a_{-i}), \beta_i(a_{-i}))_{a_{-i} \in A'_{-i}})_{i \in I}$ as continuous functions which have δ and μ as variables and fulfill $\alpha = \beta = 0$ when $\delta = 1$ if a full rank condition is satisfied. For each $i \in I$ and $a_{-i} \in A'_{-i}$, $\left.\frac{\partial \alpha_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0}$ and $\left.\frac{\partial \beta_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0}$ are available by solving

$$\left.\frac{d\boldsymbol{W}}{d\delta}\right|_{\delta=1,\mu=0} = 0 \tag{14}$$

where $\boldsymbol{W} = ((W_i[a, \boldsymbol{V}, \sigma, 0])_{i \in I})_{a \in A'}$. Especially, if

$$\left.\frac{\partial \alpha_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0}, \left.\frac{\partial \beta_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0} < 0, \tag{15}$$

there exists $\bar{\delta}$ such that $\alpha_i(a_{-i}), \beta_i(a_{-i}) \in (0,1)$ when $\delta > \bar{\delta}$ and $\mu = 0$. By calculating (14), we obtain (1) where $\lambda_i(a) = \left.\frac{\partial \alpha_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0}$ if $a_i = C_i$ and $\lambda_i(a) = \left.\frac{\partial \beta_i(a_{-i})}{\partial \delta}\right|_{\delta=1,\mu=0}$ if $a_i = D_i$. Since (1) has a unique solution, a full rank condition holds. Then, (15) hold if and only if the solutions of (1) satisfies $\boldsymbol{\lambda} \ll 0$. Thus, we have

$$^{\exists}\bar{\delta} < 1 \quad ^{\forall}\delta \in (\bar{\delta}, 1) \quad \alpha_i, \beta_i \in (0,1)$$

for $\mu = 0$. Then, we can say

$$^{\forall}\delta \in (\bar{\delta}, 1) \quad ^{\exists}\bar{\mu} \gg 0 \quad ^{\forall}\mu(0 \ll \mu \ll \bar{\mu}) \quad \alpha_i, \beta_i \in (0,1)$$

from continuity. □

Appendix C : The Proof of Lemma 2

Take $\{a_{-i}(k)\}_{k=1}^{2^{N-1}}$ and $\{\tilde{a}_{-i}(k)\}_{k=1}^{2^{N-1}}$ satisfying

$$\lambda_i(C_i, a_{-i}(k+1)) < \lambda_i(C_i, a_{-i}(k)) < 0 \tag{16}$$

and

$$\lambda_i(D_i, \tilde{a}_{-i}(k+1)) < \lambda_i(D_i, \tilde{a}_{-i}(k)) < 0 \tag{17}$$

where $\boldsymbol{\lambda}$ is defined by substituting $\boldsymbol{v} = \boldsymbol{V}$ into (1). [11] Take random events according to Condition 3.

We denote by $F(\tau, T, r)$ the probability that $\psi_i^C(k)$ occurs r times during the first T periods when $(C_i, a_{-i}(k+1))$ is played until period τ and $(C_i, a_{-i}(k))$ is played after period τ.

We say $F(\tau, T, r)$ is *single-peaked with respect to* r if there exists a non-negative integer $\bar{r}$ satisfying $F(\tau, T, r) \leq F(\tau, T, r+1)$ when $r < \bar{r}$ and $F(\tau, T, r) \geq F(\tau, T, r+1)$ when $r \geq \bar{r}$. Similarly, $F(\tau, T, r)$ is *single-peaked with respect to* τ if there exists a non-negative integer $\bar{\tau}$ satisfying $F(\tau, T, r) \leq F(\tau+1, T, r)$ when $\tau < \bar{\tau}$ and $F(\tau, T, r) \geq F(\tau+1, T, r)$ when $\tau \geq \bar{\tau}$.

The following Lemmas 3, 4 and 5 are the preliminary results. Lemma 3 is the same as Lemma A1 in Matsushima [14] and Lemma 4 is the same as Lemma 2 in Matsushima [11].

Lemma 3 : *For each $z > 0$, there exists an infinite sequence of integers $\{Z_T\}_{T=1}^{\infty}$ satisfying $T\underline{q} < Z_T < T\overline{q}$,*

$$\lim_{T\to\infty} \sum_{r=0}^{Z_T} F(0, T, r) = 1, \tag{18}$$

$$\lim_{T\to\infty} \frac{Z_T}{T} = \underline{q} \tag{19}$$

and

$$\lim_{T\to\infty} TF(0, T-1, Z_T) > z. \tag{20}$$

Lemma 4 : *$F(\tau, T, r)$ is single-peaked with respect to r and τ.*

Lemma 5 : *Fix a positive integer x. Then, for all q satisfying $0 < q < 1$ and $q \neq \underline{q}$,*

$$\lim_{T\to\infty} T^x F(0, T-1, [(T-1)q]) = 0.$$

Also, for all $0 < q < 1$,

$$\lim_{T\to\infty} T^x F([(T-1)q], T-1, Z_T) = 0.$$

Here, $[\cdot]$ means Gauss sign.

---

[11] When $\lambda_i(C_i, a_{-i}) = \lambda_i(D_i, a_{-i})$ for some $i \in I$ and $a_{-i} \in A_{-i}$, we need some modifications to the strategy because α and β may be negative. Modification is as follows; player i who takes D_i in the current phase chooses C_i and D_i with probability β_i^+ and $1 - \beta_i^+$ respectively if no events have occured in the last T periods where β_i^+ is small enough.

proof : First, we will show that $T^xF(0,T-1,[(T-1)q]) \to 0$. Note that

$$\begin{aligned} F(0,T-1,[(T-1)q]) =& {}_{T-1}C_{[q(T-1)]}\{\underline{q}\}^{[q(T-1)]}\{1-\underline{q}\}^{T-1-[q(T-1)]} \\ <& {}_{T-1}C_{[q(T-1)]}\{\underline{q}\}^{q(T-1)-1}\{1-\underline{q}\}^{T-2-q(T-1)}. \end{aligned} \tag{21}$$

Here, ${}_xC_y$ is defined by $\frac{x!}{y!(x-y)!}$. Define

$$f(T-1) = T^x_{T-1}C_{[q(T-1)]}\{\underline{q}\}^{q(T-1)-1}\{1-\underline{q}\}^{T-2-q(T-1)}.$$

Then, we have

$$\frac{f(T)}{f(T-1)} = \left(\frac{T+1}{T}\right)^x \frac{T!\,([q(T-1)])!\,(T-1-[q(T-1)])!}{([qT])!\,(T-[qT])!}\{\underline{q}\}^q\{1-\underline{q}\}^{1-q}. \tag{22}$$

Note that

$$\frac{T!\,([q(T-1)])!\,(T-1-[q(T-1)])!}{([qT])!\,(T-[qT])!} = \begin{cases} \frac{T}{T-[qT]} & \textit{if} \quad [q(T-1)] = [qT] \\ \frac{T}{[qT]} & \textit{if} \quad [q(T-1)] = [qT]-1 \end{cases}. \tag{23}$$

When q is a rational number, we can write $q = \frac{b}{a}$ where a and b are integers. Since $\frac{[qT]}{T} \to q$, it follows from (22) and (23) that

$$\lim_{T\to\infty} \frac{f(T+a)}{f(T)} = \left(\frac{a}{b}\right)^b \left(\frac{a}{a-b}\right)^{a-b} \{\underline{q}\}^b\{1-\underline{q}\}^{a-b}.$$

The first order condition of the logarithm of the right-hand side is

$$\log\left(\frac{a(1-\underline{q})}{a-b}\right) = 0$$

when we regard a as a variable. This implies that $\lim_{T\to\infty}\frac{f(T+a)}{f(T)}$ is maximized when $a = \frac{b}{\underline{q}}$, meaning $q = \underline{q}$. Since the maximized value is one, we can take a positive number $M < 1$ such that $\lim_{T\to\infty}\frac{f(T+a)}{f(T)} \to M$. This and (21) imply that

$$\lim_{T\to\infty} T^xF(0,T-1,[(T-1)q]) \to 0.$$

When q is a irrational number, we can approximate q by a rational number q' and obtain $T^xF(0,T-1,[(T-1)q]) \to 0$.

Next we will show that $\lim_{T\to\infty} T^xF([q(T-1)],T-1,Z_T) = 0$. From the single-peakedness of F, we have

$$\begin{aligned} & T^xF([q(T-1)],T-1,Z_T) \\ =& T^x\left(\sum_{l=0}^{Z_T} F([q(T-1)],[q(T-1)],l)F(0,T-1-[q(T-1)],Z_T-l)\right) \\ <& T^x\left(\sum_{l=0}^{K} F([q(T-1)],[q(T-1)],K)F(0,T-1-[q(T-1)],Z_T-l)\right) \\ & + T^x\left(\sum_{l=K+1}^{Z_T} F([q(T-1)],[q(T-1)],l)F(0,T-1-[q(T-1)],Z_T-K)\right) \\ <& T^{x+1}\left(F([q(T-1)],[q(T-1)],K) + F(0,T-1-[q(T-1)],Z_T-K)\right) \\ <& T^{x+1}(F([q(T-1)],[q(T-1)],K) + F(0,T-1-[q(T-1)],p(T-1-[q(T-1)]))) \to 0 \end{aligned}$$

for sufficiently large T where $K = \left[\frac{\bar{q}-\underline{q}}{2}\left[q(T-1)\right]\right]$ and p satisfies $0 < p < \underline{q}$ and $p(T-1-[q(T-1)]) > Z_T - K$. $\square$

Take z large enough and choose Z_T according to Lemma 3.

Let $\hat{a}_i(T)$ represent the sequence $\{a_i^t\}_{t=1}^T$ satisfying $a_i^t \in A_i$ for all t. Given $\sigma_i^T \in \Sigma_i^T$, we denote by $s_i(\hat{a}_i(T))$ the strategy for player i which urges her to play a_i^t in period t where $1 \leq t \leq T$ and to perform a review strategy σ^T which begins with C_i after period T. Given $\sigma^T \in \Sigma^T$, $\tilde{a}_{-i} \in A'_{-i}$, $\hat{a}_i(T) \in (A_i)^T$ and $a_{-i} \in A'_{-i}$, we denote the probability that $\tilde{a}_{-i}$ is played in the next phase when the opponents perform $\sigma_{-i}^T(a_{-i})$ and $s_i(\hat{a}(T))$ is played by $\zeta^T(\tilde{a}_{-i}, \hat{a}_i(T), a_{-i}, \sigma^T)$. Let $E_i^T[\boldsymbol{V}|\hat{a}_i(T), a_{-i}, \sigma^T]$ represent the expected value when $V_i(\tilde{a}_{-i})$ is obtained with probability $\zeta^T(\tilde{a}_{-i}, \hat{a}_i(T), a_{-i}, \sigma^T)$. Define

$$W_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T) = (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\pi_i(a_i^t, a_{-i}) + \delta^T E_i^T[\boldsymbol{V}|\hat{a}_i(T), a_{-i}, \sigma^T].$$

In particular, if $a_i^t = a_i$ for all t, let

$$E_i^T[\boldsymbol{V}|a, \sigma^T] = E_i^T[\boldsymbol{V}|\hat{a}_i(T), a_{-i}, \sigma^T]$$

and

$$W_i^T(a, \boldsymbol{V}, \sigma^T) = W_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T)$$

for all $a_i \in A_i$. Then, we can say σ^T is the review strategy which generates $\boldsymbol{V}$ if

$$W_i^T(a, \boldsymbol{V}, \sigma^T) = V_i(a_{-i}) \tag{24}$$

for all $i \in I$ and $a \in A'$.

Let α_i^+ be the function which has δ^T as a variable and satisfies $\alpha_i^+ = 0$ when $\delta^T = 1$. Fix $-\frac{\partial \alpha_i^+}{\partial \delta^T}$ large enough. Then, equalities (24) hold for all $i \in I$ and $a \in A'$ when $\delta^T = 1$ and $\alpha_i(k) = \beta_i(k) = 0$. From the implicit function theorem, we can also satisfy (24) when δ^T is close to unity by regarding $\alpha_i(k)$ and $\beta_i(k)$ as the functions which have δ^T as a variable and fuifill $\alpha_i(k) = \beta_i(k) = 0$ with $\delta^T = 1$. From (18), (19) and the law of large numbers, we have

$$\lim_{(T,\delta^T)\to(\infty,\delta^*)} \alpha_i(k) = \begin{cases} \alpha_i(a_{-i}(k+1)) - \alpha_i(a_{-i}(k) & \text{if} \quad k < 2^{N-1} \\ \alpha_i^+ - \alpha_i(a_{-i}(2^{N-1})) & \text{if} \quad k = 2^{N-1} \end{cases}$$

and

$$\lim_{(T,\delta^T)\to(\infty,\delta^*)} \beta_i(k) = \begin{cases} \beta_i(\tilde{a}_{-i}(k)) - \beta_i(\tilde{a}_{-i}(k-1)) & \text{if} \quad k > 1 \\ \beta_i(\tilde{a}_{-i}(1)) & \text{if} \quad k = 1 \end{cases}$$

where $\alpha_i(a_{-i})$ and $\beta_i(a_{-i})$ satisfy (13) with $(\delta, \mu) = (\delta^*, 0)$. This fact implies that

$$\lim_{T\to\infty} \left.\frac{\partial \alpha_i(k)}{\partial \delta^T}\right|_{\delta^T=1} = \begin{cases} \lambda_i(C_i, a_{-i}(k+1)) - \lambda_i(C_i, a_{-i}(k)) & \text{if} \quad k < 2^{N-1} \\ \frac{\partial \alpha_i^+}{\partial \delta^T} - \lambda_i(C_i, a_{-i}(2^{N-1})) & \text{if} \quad k = 2^{N-1} \end{cases}$$

and

$$\lim_{T\to\infty} \left.\frac{\partial \beta_i(k)}{\partial \delta^T}\right|_{\delta^T=1} = \begin{cases} \lambda_i(D_i, \tilde{a}_{-i}(k)) - \lambda_i(D_i, \tilde{a}_{-i}(k-1)) & \text{if} \quad k > 1 \\ \lambda_i(D_i, \tilde{a}_{-i}(1)) & \text{if} \quad k = 1 \end{cases}.$$

Then, from (16) and (17), it follows that

$$
\begin{aligned}
&{}^{\exists}T^* \quad {}^{\forall}T > T^* \quad {}^{\exists}\overline{\delta} \quad {}^{\forall}\delta > \overline{\delta} \\
&0 < \alpha_i^+ < 1, \quad 0 < \alpha_i(k) < 1, \quad \sum_{k=1}^{2^{N-1}} \alpha_i(k) < \alpha_i^+, \\
&0 < \beta_i(k) < 1, \quad \sum_{k=1}^{2^{N-1}} \beta_i(k) < 1.
\end{aligned}
$$

Hence, we can say

$$
{}^{\exists}T^* \quad {}^{\forall}T > T^* \quad {}^{\exists}\overline{\delta} \quad {}^{\forall}\delta > \overline{\delta} \quad (\textit{there exists } \sigma^T \textit{ which generates } \boldsymbol{V}). \tag{25}
$$

Subsequently, we show that σ^T is a sequential equilibrium. Since Condition 3 implies that private signal ω_i has no information on the occurance of opponents' random events, it suffices to show that deviations with a history independent strategy does not bring him profits.

Let $\xi(\hat{a}_i(T), a_{-i}, X_j)$ represent the probability that event X_j occurs in the first phase when he performs $s_i(\hat{a}_i(T))$ and the opponents play $(\sigma_j^T(a_j))_{j \neq i}$. Note that Condition 3 implies that player i's random event and player j's random event are independent. Then, by calculation, we have

$$
\begin{aligned}
&\left.\frac{dW_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} \\
=& -\sum_{t=1}^{T} \frac{\pi_i(a_i^t, a_{-i})}{T} \\
&+ \sum_{j \neq i (a_j = C_j)} (V_i(a_{-i}) - V_i(a_{-(i,j)}, D_j)) \left(\sum_{l=1}^{2^{N-1}-1} \xi(\hat{a}_i(T), a_{-i}, X_j(C,l)) \left.\frac{\partial \alpha_j(l)}{\partial \delta^T}\right|_{\delta^T=1} \right) \\
&+ \sum_{j \neq i (a_j = C_j)} (V_i(a_{-i}) - V_i(a_{-(i,j)}, D_j)) \xi(\hat{a}_i(T), a_{-i}, X_j^+) \left.\frac{\partial \alpha_j(2^{N-1})}{\partial \delta^T}\right|_{\delta^T=1} \\
&+ \sum_{j \neq i (a_j = D_j)} (V_i(C_j, a_{-(i,j)}) - V_i(a_{-i})) \left(\sum_{l=2}^{2^{N-1}} \xi(\hat{a}_i(T), a_{-i}, X_j(D,l)) \left.\frac{\partial \beta_i(l)}{\partial \delta^T}\right|_{\delta^T=1} \right) \\
&+ \sum_{j \neq i (a_j = D_j)} (V_i(C_j, a_{-(i,j)}) - V_i(a_{-i})) \xi(\hat{a}_i(T), a_{-i}, X_j^{++}) \left.\frac{\partial \beta_i(1)}{\partial \delta^T}\right|_{\delta^T=1}
\end{aligned} \tag{26}
$$

for all $a_{-i} \in A'_{-i}$. Define

$$
W_i(a_{-i}, \tau, T) = W_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T) - W_i^T(C_i, a_{-i}, \boldsymbol{V}, \sigma^T)
$$

where $a_i^t = D_i$ for all $t \in \{1, \ldots, \tau\}$ and $a_i^t = C_i$ for all $t \in \{\tau+1, \ldots, T\}$.

Lemma 6 : *For all* $\tau \in \{1, \ldots, T-1\}$ *and* $a_{-i} \in A'_{-i}$, $\left.\frac{dW_i(a_{-i}, \tau, T)}{d\delta^T}\right|_{\delta^T=1} > 0$ *when* T *is large enough.*

proof : We deal with only the case where $\pi_i(C_i, a_{-i}) < \pi_i(D_i, a_{-i})$ and there exist player j and player k satisfying $a_j = C_j$ and $a_k = D_k$. We can prove the other cases in the same way.

Take integers $n_j(C)$ and $n_j(D)$ for each $j \neq i$ satisfying $a_{-j}(n_j(C)) = (C_i, a_{-i})$ and $a_{-j}(n_j(D)) = (D_i, a_{-i})$ if $a_j = C_j$ and $\tilde{a}_{-j}(n_j(C)) = (C_i, a_{-i})$ and $\tilde{a}_{-j}(n_j(D)) = (D_i, a_{-i})$ if $a_j = D_j$. Let $I(C)$ denote the set of players satisfying $a_j = C_j$ and $n_j(C) < n_j(D)$. Similarly, $I'(C)$ denote the set of players satisfying $a_j = C_j$ and $n_j(C) > n_j(D)$. Let $I(D)$ represent the set of players satisfying $a_j = D_j$ and $n_j(C) > n_j(D)$. Let $I'(D)$ represent the set of players satisfying $a_j = D_j$ and $n_j(C) < n_j(D)$. Define

$$\gamma_j = \begin{cases} \sum_{k=n_j(C)}^{n_j(D)-1} \frac{\partial \alpha_j(k)}{\partial \delta^T}\Big|_{\delta^T=1} & \text{if} \quad j \in I(C) \\ -\sum_{k=n_j(D)}^{n_j(C)-1} \frac{\partial \alpha_j(k)}{\partial \delta^T}\Big|_{\delta^T=1} & \text{if} \quad j \in I'(C) \\ \sum_{k=n_j(D)+1}^{n_j(C)} \frac{\partial \beta_j(k)}{\partial \delta^T}\Big|_{\delta^T=1} & \text{if} \quad j \in I(D) \\ -\sum_{k=n_j(C)+1}^{n_j(D)} \frac{\partial \beta_j(k)}{\partial \delta^T}\Big|_{\delta^T=1} & \text{if} \quad j \in I'(D) \end{cases}.$$

From (26),

$$\begin{aligned} &\frac{dW_i(a_{-i},\tau,T)}{d\delta^T}\Big|_{\delta^T=1} \\ =&\frac{\tau(\pi_i(C_i,a_{-i}) - \pi_i(D_i,a_{-i}))}{T} \\ &- \sum_{j\in I(C)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})\sum_{k=0}^{\tau-1} F(k,T-1,Z_T) \\ &- \sum_{j\in I(D)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})\sum_{k=0}^{\tau-1} F(k,T-1,Z_T) \\ &- \sum_{j\in I'(C)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})\sum_{k=0}^{\tau-1} F(T-1-k,T-1,Z_T) \\ &- \sum_{j\in I'(D)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})\sum_{k=0}^{\tau-1} F(T-1-k,T-1,Z_T). \end{aligned} \tag{27}$$

Define

$$\begin{aligned} F^*(k) = &\sum_{j\in I(C)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})F(k,T-1,Z_T) \\ &+ \sum_{j\in I(D)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})F(k,T-1,Z_T) \\ &+ \sum_{j\in I'(C)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})F(T-1-k,T-1,Z_T) \\ &+ \sum_{j\in I'(D)} (V_i(C_j,a_{-(i,j)}) - V_i(C_j,a_{-(i,j)}))\gamma_j(\overline{q}-\underline{q})F(T-1-k,T-1,Z_T). \end{aligned}$$

Note that $I(C) \neq \emptyset$ or $I(D) \neq \emptyset$ since (24) holds and $\pi_i(C_i, a_{-i}) < \pi_i(D_i, a_{-i})$. Then, from (20) and Lemmas 4 and 5, there exists an integer k_T such that $-TF^*(k) > \pi_i(D_i,a_{-i}) - \pi_i(C_i,a_{-i})$ if $k < k_T$ and $-TF^*(k) \leq \pi_i(D_i,a_{-i}) - \pi_i(C_i,a_{-i})$ if $k \geq k_T$ for sufficiently large T, since z is large enough. Note that $\frac{dW_i(a_{-i},T,T)}{d\delta^T}\Big|_{\delta^T=1} = 0$ from the property of σ^T. Hence, it follows from (27) that $\frac{dW_i(a_{-i},\tau,T)}{d\delta^T}\Big|_{\delta^T=1} > 0$ for all $1 \leq \tau \leq T-1$. $\square$

Note that $W_i(\underline{a}_{-i}, \tau, T) = 0$ when $\delta^T = 1$ and $\alpha = \beta = 0$. Hence, Lemma 6 implies that there exists $\overline{T}$ such that for each $T > \overline{T}$ there exists $\overline{\delta}$ satisfying $W_i(a_{-i}, \tau, T) < 0$ for all $\delta > \overline{\delta}$. Define

$$W_i'(a_{-i}, \tau, T) = W_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T) - W_i^T(C_i, a_{-i}, \boldsymbol{V}, \sigma^T)$$

where $a_i^t = C_i$ for all $t \in \{1, \ldots, \tau\}$ and $a_i^t = D_i$ for all $t \in \{\tau+1, \ldots, T\}$. Then, we can easily see that

$$\left.\frac{dW_i(a_{-i}, \tau, T)}{d\delta^T}\right|_{\delta^T=1} = \left.\frac{dW_i'(a_{-i}, T-\tau, T)}{d\delta^T}\right|_{\delta^T=1}.$$

Therefore, deviations between C_i and D_i doesn't bring player i profits.

Subsequently, we'll show that playing $a_i \notin A_i'$ is not profitable for player i. First, we can easily see that $\left.\frac{dW_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} > 0$ if $a_i^t \in A_i \setminus \{C_i\}$ for all t or $a_i^t \in A_i \setminus \{D_i\}$ for all t in the same way as Lemma 6. Fix $\hat{a}_i(T)$ satisfying $a_i^t \notin A_i'$ for some t and let $\hat{T}$ and $\tilde{T}$ be the set of t satisfying $a_i^t \notin A_i'$ and $a_i^t = D_i$, respectively. Fix $\hat{a}_i'(T)$ satisfying $a_i'^t \notin A_i'$ for all $t \in \tilde{T}$ and $a_i'^t = a_i^t$ for all $t \notin \tilde{T}$. Fix $\hat{a}_i''(T)$ satisfying $a_i''^t = D_i$ for all $t \in \hat{T}$ and $a_i''^t = a_i^t$ for all $t \notin \hat{T}$. Then, we can show that either

$$\left.\frac{dW_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} - \left.\frac{dW_i^T(\hat{a}_i'(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} > 0$$

or

$$\left.\frac{dW_i^T(\hat{a}_i(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} - \left.\frac{dW_i^T(\hat{a}_i''(T), a_{-i}, \boldsymbol{V}, \sigma^T)}{d\delta^T}\right|_{\delta^T=1} > 0.$$

From the above argument, it follows that

$$^{\exists}\overline{T} \quad ^{\forall}T > \overline{T} \quad ^{\exists}\overline{\delta} \quad ^{\forall}\delta > \overline{\delta} \quad (\sigma^T \textit{ is a sequential equilibrium}).$$

Then, from (25),

$$^{\exists}\overline{T} \quad ^{\forall}T > \overline{T} \quad ^{\exists}\overline{\delta} \quad ^{\forall}\delta > \overline{\delta}$$
$$(\textit{ there exists the sequential equlibrium } \sigma^T \textit{ which sustains } \boldsymbol{V})$$

and we have finished the proof. □

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