On the Role of Tax-Subsidy Scheme in Money Search Models*

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Abstract

This paper investigates the role of policy in money search models with divisible money. Recently, real indeterminacy of stationary equilibria has been found in both specific and general search models with divisible money. Thus if we assume the divisibility of money, it would be quite difficult to make accurate predictions of the effects of simple monetary policies. Instead, we show that some tax-subsidy schemes select a determinate efficient equilibrium. In other words, for a given efficient equilibrium and for any real number $\delta > 0$, a certain tax-subsidy scheme induces a locally determinate equilibrium within the δ-neighborhood of the given equilibrium. Moreover, the size of the tax-subsidy can be arbitrarily small.

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1 Introduction

This paper studies the roles of tax-subsidy schemes in money search models. In most of cases, money is indivisible and the stationary equilibria are determinate. Thus the effects of the policies are determinate as well. However, real indeterminacy of stationary equilibria has been recently found in both specific and general search models with divisible money. (See, for example, Green and Zhou [3] [4], Kamiya and Shimizu [6], Matsui and Shimizu [7], and Zhou [9].) In other words, if we assume the divisibility of money in these models, the stationary equilibria become indeterminate. Thus it is quite difficult to make accurate predictions of the effects of simple monetary policies in such models. Instead, we show that some tax-subsidy schemes select a determinate efficient equilibrium. In other words, for a given efficient equilibrium and for any real number

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$\delta > 0$, a certain tax-subsidy scheme induces a locally determinate equilibrium within the δ-neighborhood of the given equilibrium. Moreover, the size of the tax-subsidy can be arbitrarily small.

In order to explain the indeterminacy of equilibria, we first consider a random matching model with divisible money and without intervention of government. There is a continuum of private agents who meets pairwise according to a random matching process. In each meeting, there is no double coincidence of wants, and therefore fiat money can be used as a medium of exchange. In such a model, the conditions for a stationary equilibrium are (i) each agent maximizes the expected value of utility-streams, i.e., the Bellman equation is satisfied, and (ii) the money holdings distribution of the economy is stationary, i.e., time-invariant.

In some special models with divisible money, Green and Zhou [3], Matsui and Shimizu [7], and Zhou [9] found indeterminacy of stationary equilibria by calculating explicit solutions. Kamiya and Shimizu [6] found the underlying logic of indeterminacy; namely, there is at least one-degree of freedom in the condition for the stationarity of money holdings distributions.

To be more precise, we focus on a stationary equilibrium in which all transactions are made with integer multiples of some $p > 0$ and money holdings distributions have a support expressed by $\{0, p, 2p, \ldots, Np\}$ for some positive integer N. Let $h = (h_0, h_1, \ldots, h_N)$ be a probability distribution on the support, where h_n is a measure of agent with money holding np. Suppose the values of the other variables, besides h, are given. Note that these variables are determined by the Bellman equations for a given h; namely, the number of these variables are equal to the number of equations in the Bellman equations. Let I_n, a function of h, be a measure of agents whose money holdings are not np before trades and become np after trades, and O_n, a function of h, be a measure of agents whose money holdings are np before trades and become $n'p$ for some $n' \neq n$ after trades. In other words, I_n is the measure of agents in the inflow at np, and O_n is the measure of agents in the outflow at np. The stationary condition is expressed by $I_n = O_n$ for $n = 0, 1, \ldots, N$.

Clearly, $\sum_{n=0}^{N} I_n - \sum_{n=0}^{N} O_n = 0$ always holds, i.e., this is an identity, since each agent, who belongs to an outflow at some n, should belong to an inflow at some n' and thus the total measure of agents in all inflows, expressed by $\sum_{n=0}^{N} I_n$, is equal to that in all outflows, expressed by $\sum_{n=0}^{N} O_n$. On the other hand, $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ always holds, i.e., this is also an identity. To see this, suppose that two agents, say a buyer and a seller, meet and a monetary trade occurs. Then the amount of money the buyer pays is equal to that of the seller obtains; in other words, the sum of their money holdings before trade is equal to that of after trade. Since this holds in each trade, the total amount of money before trades, expressed by $\sum_{n=0}^{N} pnO_n$, is equal to the total amount of money after trades, expressed by $\sum_{n=0}^{N} pnI_n$, and thus $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ always holds. Thus there are $N+1$ equations, $I_n = O_n$, $n = 0, 1, \ldots, N$, and two identities, $\sum_{n=0}^{N} I_n - \sum_{n=0}^{N} O_n = 0$ and $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$; i.e., the number of linearly independent equations among them is $N - 1$. On the other hand, $\sum_{n=0}^{N} h_n - 1 = 0$ is the other restriction and the number of variables, $h_0, h_1, \ldots, h_N$, is $N + 1$. Therefore there is at least one degree of freedom in the determination of stationary distribution. This leads to the real indeterminacy of stationary equilibria in random matching models with divisible money satisfying some regularity conditions. For the details, see Kamiya and Shimizu [6].

Now we introduce a policy into this economy. More precisely, following Aiyagari et al. [1], we introduce government agents who are similar to private agents in terms of pairwise matching. According to the rule called a tax-subsidy scheme which is enforced by the government, they collect tax from or give subsidy to matched private agents. If the total amount of tax is more than that of subsidy, the monetary authority absorbs the surplus, while if the total amount of tax is less than that of subsidy, the authority issues the necessary amount. Thus the total amount of money the private agents have at the beginning of the period is not necessarily equal to that at the end of the period. This implies that $\sum_{n=0}^{N} nI_n - \sum_{n=0}^{N} nO_n = 0$ does not always hold. Thus the total number of equations is equal to that of variables and the stationary equilibria become determinate.

Furthermore, we show that a tax-subsidy scheme can select a determinate efficient equilibrium. In other words, for a given equilibrium and for any real number $\delta > 0$, a certain tax-subsidy scheme induces a locally determinate equilibrium within the δ-neighborhood of the given equilibrium. Of course, the given equilibrium can be an efficient one, i.e., an equilibrium with high welfare. For that purpose, the size of the tax-subsidy can be arbitrarily small.

It is well-known that, in the standard general equilibrium model, equilibria are generically determinate and the lump-sum tax-subsidy only leads the economy to a given Pareto efficient equilibrium. On the other hand, in the money search model with divisible money, equilibria are indeterminate and the tax-subsidy scheme has another role; namely, it makes an efficient equilibrium determinate. In other words, we have found a new role of tax-subsidy schemes which has not yet been known so far in the literature.

The plan of this paper is as follows. In Section 2, we investigate a special model which can be considered as Zhou [9]'s model with government agents. In Section 3, we present a general model, to which most of random matching models with divisible money belong, and investigate tax-subsidy schemes focusing on pure strategy equilibria. Then in Section 4, we extend the results in Section 3 to the case of mixed strategy equilibria. Finally, in Section 5, we conclude the paper with some discussion.

2 A Model with Government Agents

2.1 Model and Definitions

We first present a simple model with government agents. Our model can be considered as Zhou [9]'s model with government agents introduced by Aiyagari et al. [1].

There is a continuum of private agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A type $i-1$ agent can produce just one unit of type i good and the production cost is $c > 0$. (We assume that a type k agent produces type 1 good.) A type i agent obtains utility $u > 0$ only when she consumes one unit of type i good. We assume $u > c$. Time is continuous and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$. For every matched pair, the seller posts a take-it-or-leave-it price offer without knowing the amount of the buyer's money holdings. Let $M > 0$ be the nominal stock of fiat money, and $\gamma > 0$ be the discount rate.

In what follows, we focus on a stationary distribution of money holdings of the private agents with the support $\{0, p, \dots\}$ for some $p > 0$. Thus the money holdings distribution can be expressed by $h_n, n = 0, 1, \dots,$ the measure of the set of private agents with money holding np. Of course, h satisfies $\sum_n h_n = 1$ and $h_n \geq 0$ for all n.

We introduce government agents to this economy. They are "programmed" to follow the rule specified later. That is, following the given rule, they collect tax from or give subsidy to the agents they are matched with. We assume that government agents can observe current money holdings of agents they are matched with. Let $G > 0$ be the measure of the government agents. Thus the total measure of all agents is $1 + G$. Note that in the following arguments G can be any small positive number. We describe the policy of the government by $(t_0, t_1, \dots)$, where $t_n \in [-1, 1]$. When a government agent meets an agent with $\eta \in [np, (n+1)p)$, she gives subsidy p with probability $|t_n|$ if $t_n > 0$, while she collects tax p with probability $|t_n|$ if $t_n < 0$. If the total amount of tax is more than that of subsidy, the government absorbs the surplus, while if the total amount of tax is less than that of subsidy, the government issues the necessary amount.

We focus on stationary equilibria in which all agents with identical characteristics act similar and in which all of the k types are symmetric. Since relevant decisions for an agent are only what price she offers to a buyer of her production good, and how to respond to an offer made by a seller of her consumption good, then we consider her strategy as a pair of functions of money holdings: $\omega(\eta) \in \mathbb{R}_+ \cup \{NT\}$, an offer price, and $\rho(\eta) \in \mathbb{R}_+$, a reservation price, when her money holdings is $\eta \in \mathbb{R}_+$.[1] Here, $\omega(\eta) = NT$ implies that she rejects a trade. Since the reservation price cannot exceed the buyer's money holdings, ρ should satisfy the following feasibility condition:

$$\rho(\eta) \leq \eta. \tag{1}$$

We adopt one type of the Bayesian perfect equilibrium, called a stationary equilibrium, as our equilibrium concept. Since the rigorous definition is rather complicated, then we present it in Appendix A. Instead, in Theorem 1 in the next subsection, we present the conditions for stationary equilibria in the case with the following strategy. Namely, we restrict our attention to a stationary equilibrium with the following strategy both in the cases with and without tax-subsidy: there exists a positive integer N such that

- a seller with η, $0 \leq \eta < Np$, offers p,
- a seller with η, $\eta \geq Np$, offers NT, and
- the reservation price of a buyer with η, $\eta \geq p$, is more than or equal to p.

Note that if the above strategy is indeed an equilibrium, then on the equilibrium path, all trades occur with $p > 0$. Moreover, $\eta > Np$ does not occur in the stationary distribution; i.e., any $n > N$ is a transient state and N is the endogenously determined upper bound of money holdings. Thus $\{0, p, \dots, Np\}$ can be the support of a stationary distribution with the strategy. In what follows, a stationary equilibrium in which all trades occur with a single price is called a single price equilibrium (SPE).

[1] Throughout this section we focus on equilibria with pure strategies.

2.2 SPE without Tax-Subsidy

First, we consider the case that $t_n = 0$ for all n, i.e., the case without tax-subsidy. According to the strategy specified above, an agent moves out from np either by making a sale or by making a purchase. More precisely, A type i agent with $np < Np$ makes a sale when she meets a type $i+1$ agent with money. The measure of agents with np is h_n and the probability that they can make a sale is $\frac{\mu\kappa}{1+G}(1-h_0)$, and thus the set of agents with measure $\frac{\mu\kappa}{1+G}h_n(1-h_0)$ moves out from np, i.e., it is an outflow at np as well as an inflow at $(n+1)p$. On the other hand, a type i agent with $np > 0$ makes a purchase when she meets a type $i-1$ agent with $np < Np$. The measure of agents with np is h_n and the probability that they can make a purchase is $\frac{\mu\kappa}{1+G}(1-h_N)$, and thus the set of agents with measure $\frac{\mu\kappa}{1+G}h_n(1-h_N)$ moves out from np, i.e., it is an outflow at np as well as an inflow at $(n-1)p$. Thus the stationary condition for $h = (h_0, h_1, \ldots, h_N)$, i.e., the time rate of inflow at n is equal to the time rate of outflow at n, is expresses as follows:

$$\frac{\mu\kappa}{1+G}\left[h_1(1-h_N) - h_0(1-h_0)\right] = 0, \tag{2}$$

$$\frac{\mu\kappa}{1+G}\left[\{h_{n-1}(1-h_0) + h_{n+1}(1-h_N)\} - h_n\{(1-h_0)+(1-h_N)\}\right] = 0, \quad 1 \le n \le N-1, \tag{3}$$

$$\frac{\mu\kappa}{1+G}\left[h_{N-1}(1-h_0) - h_N(1-h_N)\right] = 0, \tag{4}$$

$$\sum_{n=0}^{N} h_n = 1. \tag{5}$$

Let the LHS of the nth equation be denoted by $D_n(h)$, the difference between the inflow into state n and the outflow from state n. It can be easily checked that $\sum_{n=0}^{N} D_n(h) = 0$ and $\sum_{n=0}^{N} nD_n(h) = 0$ always hold, i.e., they are identities. Thus if $D_n(h) = 0$ for $n = 2, \ldots, N$ hold, then $D_0(h) = 0$ and $D_1(h) = 0$ are automatically satisfied. In other words, two of the above equations are redundant. Thus the above system of equations has at least one degree of freedom.

In fact, we obtain the following stationary distribution from the stationary condition:

$$h_n = h_0\left(\frac{1-h_0}{1-h_N}\right)^n, \quad n = 1, \ldots, N, \tag{6}$$

where h_N is determined so that

$$h_N(1-h_N)^N = h_0(1-h_0)^N. \tag{7}$$

It is verified that for any $h_0 \in (0,1)$ there is the corresponding distribution h satisfying the stationary condition. In other words, we have a continuum of candidates for stationary distributions.

Let $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}_+$ be a value function. Kamiya et al. [5] show the following existence theorem for SPEs.

Theorem 1 Suppose

$$(\phi+1)^N < \frac{u}{c} < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}, \tag{8}$$

holds for a positive integer N, where $\phi = \frac{(1+G)\gamma}{\mu\kappa}$.[2] [3] Then, for some $\epsilon > 0$, there exists $(h, \mathcal{V})$ such that (i) $h_0 \in (1-\epsilon, 1)$ and $h_n, n = 1, \ldots, N$, are given by (6) and (7), (ii) $\mathcal{V}$ is a solution to the Bellman equation, and (iii) the strategy specified above is the optimal policy function of the Bellman equation. For the definition of the Bellman equation, see Appendix A.

Moreover, Kamiya et al. [5] show that, in SPEs, all the relevant incentive conditions are satisfied with strict inequalities besides the boundary of the set of SPEs. Thus even if we slightly perturb the equilibrium condition by introducing a policy with small amounts of tax and subsidy, the incentive conditions still hold in most of equilibria with the policy. Of course, for this argument, we need to check the regularity condition for the implicit function theorem. It is checked in the next subsection.

Let $V_n = \mathcal{V}(np)$, $n = 0, \ldots, N$. Then we define the welfare as $W = \sum_{n=0}^{N} h_n V_n$. Then we obtain

$$W = \frac{(1-h_0)(1-h_N)}{\phi}(u-c).$$

It is verified that W takes a value in $\Big(0, \left(\frac{N}{N+1}\right)^2 \frac{u-c}{\phi}\Big]$, and that the maximum value is attained at $h = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. Clearly, it constitutes the most efficient SPE among stationary equilibria with the upper bound of money holdings N if the incentive conditions are satisfied.

[2] This definition of ϕ is slightly different from the one defined in Kamiya et al. [5]. For the details, see Remark 2 in Appendix A.

[3] It is verified that there exists $\underline{\phi} \geq 0$ such that

$$(\phi+1)^N < \frac{\phi(\phi+1)^{2N}}{(\phi+1)^N - 1}$$

holds for $\phi > \underline{\phi}$. In other words, there exists a region of parameter profiles (ϕ, u, c) satisfying the sufficient condition. (See Kamiya et al. [5].)

2.3 SPE with Tax-Subsidy

We consider the case with tax-subsidy, i.e., t is a nonzero vector. In addition, we require that $t_0 \geq 0$ and $t_N \leq 0$ throughout the paper so that the introduction of policy does not change the support of money holdings distribution. According to the tax-subsidy scheme and the strategy specified previously, an agent moves to np either by making a sale from $(n-1)p$, by making a purchase from $(n+1)p$, by obtaining subsidy from $(n-1)p$, or by paying tax from $(n+1)p$. Thus, denoting by $\tilde{h}$ a stationary distribution with tax-subsidy, the stationary condition is as follows:

$$\frac{\mu\kappa}{1+G}\left[\left\{\tilde{h}_1(1-\tilde{h}_N)+h_1kGt_1^-\right\}-\tilde{h}_0\left\{(1-\tilde{h}_0)+kGt_0\right\}\right]=0, \tag{9}$$

$$\frac{\mu\kappa}{1+G}\Bigg[\left\{\tilde{h}_{n-1}(1-\tilde{h}_0)+\tilde{h}_{n+1}(1-\tilde{h}_N)+\tilde{h}_{n-1}kGt_{n-1}^+ +\tilde{h}_{n+1}kGt_{n+1}^-\right\} \\ -\tilde{h}_n\left\{(1-\tilde{h}_0)+(1-\tilde{h}_N)+kG|t_n|\tilde{h}_n\right\}\Bigg]=0, \quad 1\leq n\leq N-1, \tag{10}$$

$$\frac{\mu\kappa}{1+G}\left[\left\{\tilde{h}_{N-1}(1-\tilde{h}_0)+\tilde{h}_{N-1}kGt_{N-1}^+\right\}-\tilde{h}_N\left\{(1-\tilde{h}_N)-kGt_N\right\}\right]=0, \tag{11}$$

$$\sum_{n=0}^{N}\tilde{h}_n=1, \tag{12}$$

where $t_n^+ = \max\{0,t_n\}$ and $t_n^- = -\min\{0,t_n\}$. Let the nth equation be denoted by $\tilde{D}_n(\tilde{h})=0$. As in the case with tax-subsidy, $\sum_{n=0}^N \tilde{D}_n(\tilde{h})=0$ is an identity, and thus one of the above equations is redundant. On the other hand, $\sum_{n=0}^N n\tilde{D}_n(\tilde{h})=0$ is no longer an identity. Thus only one equation, say (9), is redundant and the system has no degree of freedom. Then $(h,\mathcal{V})$ is called a SPE with tax-subsidy if (10), (11), (12), and the Bellman equation are satisfied.

First, as an example, we present a method to approximate $\left(\frac{1}{N+1},\dots,\frac{1}{N+1}\right)$, which is a stationary distribution for the above strategy in the case without tax-subsidy. Suppose there exists $\mathcal{V}$ which, together with $h=\left(\frac{1}{N+1},\dots,\frac{1}{N+1}\right)$, satisfies the Bellman equation with the strict incentive condition.[4] Let $t=\epsilon\tau$ where $\epsilon>0$ and $\tau=(\tau_0,0,\dots,0,\tau_N)$, where ϵ is a size of the scheme. Let $\tau_0=-\tau_N>0$. Then we obtain a solution

$$\tilde{h}_n=\begin{cases}\tilde{h}_0 & \text{if } n=0 \text{ or } N,\\ \frac{\tilde{h}_0\left(1-\tilde{h}_0+kG\epsilon\tau_0\right)}{1-\tilde{h}_0} & \text{if } 1\leq n\leq N-1,\end{cases}$$

[4] The existence of such a $\mathcal{V}$ depends on the parameters and N.

where $\tilde{h}_0$ is a solution of the equation:

$$(N+1)(\tilde{h}_0)^2 - \{3 + (N-1)(1 + kG\epsilon\tau_0)\}\,\tilde{h}_0 + 1 = 0,$$

with $\tilde{h}_0 \in (0,1)$. It is verified that such $\tilde{h}_0$ is uniquely determined and thus $\tilde{h}$ is uniquely determined as well. Moreover, as $\epsilon \to 0$, $\tilde{h} \to \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. Note that this distribution is one of the stationary distributions without tax-subsidy and is orthogonal to τ. If the regularity condition for the implicit function theorem holds, then $\mathcal{V}$ is also a continuous function of ϵ and the incentive condition holds for a small $\epsilon > 0$. Thus we can approximate the equilibrium with $h = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. Later, we show the regularity in general.

We can generalize this method to approximate any given SPE with strict incentive condition. Note that the budget deficit is expressed as follows:

$$\frac{\mu G}{1+G}\tilde{h} \cdot t = \frac{\mu G}{1+G}\epsilon\tilde{h} \cdot \tau = \sum_{n=0}^{N} n\tilde{D}_n. \tag{13}$$

Since we do not require budget balancing, then this may not be 0 out of equilibrium. Thus $\sum_{n=0}^{N} n\tilde{D}_n = 0$ is not an identity. This enables us to make a stationary distribution determinate. Let $(h^*, \mathcal{V}^*)$ be a SPE without tax-subsidy in which the strict incentive condition holds. Then choose a τ satisfying $h^* \cdot \tau = 0$ and let $t = \epsilon\tau$. Then the equilibrium is determinate for $\epsilon > 0$, and the equilibrium for ϵ, denoted by $(\tilde{h}(\epsilon), \mathcal{V}(\epsilon))$ converges to $(h^*, \mathcal{V}^*)$ as $\epsilon \to 0$.

More precisely, in the stationarity condition (10)-(12), we can use $\tilde{h} \cdot \tau = 0$ instead of $\tilde{D}_1 = 0$. In other words, for $\epsilon > 0$, the both conditions have the same solutions because it follows from $\tilde{D}_n = 0$ for $n = 2, \ldots, N$, that $\tilde{h} \cdot \tau = 0$ imply $\tilde{D}_1 = 0$, and vice versa. We call the system of equations including $\tilde{h} \cdot \tau = 0$ the new system. Since the new system is regular at $\epsilon = 0$, h^* is a determinate solution to the new system at $\epsilon = 0$. For the regularity, see Appendix B. Thus $(\tilde{h}(\epsilon), \mathcal{V}(\epsilon))$ converges to $(h^*, \mathcal{V}^*)$ as $\epsilon \to 0$ by the implicit function theorem.

As stated in the previous subsecion, as long as we consider a small size tax-subsidy scheme, introducing the scheme just slightly perturbs the incentive conditions. Therefore if we pick up a relative interior point of the equilibrium manifold found in Theorem 1 as a goal, the incentive condition is not violated.

Note that the budget deficit is zero in the stationary distribution, since $\tilde{D}_n = 0, n = 0, \ldots, N$, hold, although it is not identically zero.

The above arguments can be summarized as follows:

Theorem 2 Suppose a SPE without tax-subsidy, in which the strategy specified in Subsection 2.1, satisfies strict incentive conditions. Then, for any $\delta > 0$, there exists a tax-subsidy scheme such that a SPE with the tax-subsidy is locally determinate and lies in the δ-neighborhood of the SPE without tax-subsidy.

In Appendix C, we present the explicit solutions, including $\mathcal{V}$ in equilibria, with and without tax-subsidy in case of $N = 1$.

2.4 Budget Balancing Rule

It is interesting to see that any tax-subsidy scheme with budget balancing rule cannot make equilibria determinate. For example, consider stationary equilibria with $N = 2$ and the tax-subsidy scheme with the form $(t_0, 0, t_2)$. It is not the case that we can freely choose both t_0 and t_2 because $t_0\tilde{h}_0 + t_2\tilde{h}_2 = 0$ must hold even out of the equilibrium path. If we set $t_2 = -1$, then $\tilde{h}$ and t_0 must satisfy

$$
\begin{aligned}
&\frac{\mu\kappa}{1+G}\left[\tilde{h}_1(1-\tilde{h}_2) - \tilde{h}_0\left(1-\tilde{h}_0+kGt_0\right)\right] = 0,\\
&\frac{\mu\kappa}{1+G}\left[\left\{\tilde{h}_0\left(1-\tilde{h}_0+kGt_0\right)+\tilde{h}_2\left(1-\tilde{h}_2+kG\right)\right\} - \tilde{h}_1\left(1-\tilde{h}_0+1-\tilde{h}_2\right)\right] = 0,\\
&\frac{\mu\kappa}{1+G}\left[\tilde{h}_1(1-\tilde{h}_0) - \tilde{h}_2\left(1-\tilde{h}_2+kG\right)\right] = 0,\\
&t_0 = \frac{\tilde{h}_2}{\tilde{h}_0},\\
&\tilde{h}_0 + \tilde{h}_1 + \tilde{h}_2 = 0.
\end{aligned}
$$

Therefore we obtain, for any $\tilde{h}_0 \in \left[\frac{4+kG-\sqrt{(4+kG)^2-12}}{6}, 1\right)$,[5]

$$
\begin{aligned}
\tilde{h}_1 &= \frac{-h_0 - kG + \sqrt{(4-3\tilde{h}_0+kG)(\tilde{h}_0+kG)}}{2},\\
\tilde{h}_2 &= \frac{2 - h_0 + kG - \sqrt{(4-3\tilde{h}_0+kG)(\tilde{h}_0+kG)}}{2}.
\end{aligned}
$$

Thus stationary distributions are indeterminate. The intuition is as follows. Requiring budget balancing, we have one more additional variable t_0, then the number of variables is one larger than that of equations. Thus there is one degree of freedom in the system of equations.

[5] The condition that $\tilde{h}_0 \geq \frac{4+kG-\sqrt{(4+kG)^2-12}}{6}$ is necessary for $t_0 \leq 1$.

3 A General Model

In this section, we show the same results for a general model as in the previous section. The private sector is slightly a special case of the one investigated by Kamiya and Shimizu [6] (hereafter, we call KS simply).

There is a continuum of private agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A type i good is produced by a type $i-1$ agent. A type i agent obtains some positive utility only when she consumes type i good. We make no assumption on the divisibility of goods. We assume that fiat money is durable and perfectly divisible. Time is continuous, and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$.

We confine our attention to the case that, for some positive number p, all trades occur with its integer multiple amounts of money. In what follows, we focus on a stationary distribution of economy-wide money holdings on $\{0, \ldots, N\}$ expressed by $h = (h_0, \ldots, h_N)$, where h_n is the measure of agents with np amount of money, and $N < \infty$ is the upper bound of the distribution. Our model includes the case of exogenously determined N as well as the case of endogenously determined N. Of course, $h_n \geq 0$ and $\sum_{n=0}^{N} h_n = 1$ hold. Let $M > 0$ be a given nominal stock of money circulating in the private sector. Since p is uniquely determined by $\sum_{n=0}^{N} pnh_n = M$ for a given h for $h_0 \neq 1$, then, deleting p from $\{0, p, \ldots, Np\}$, the set $\{0, \ldots, N\}$ can be considered as the state space.

Since we adopt a general framework, then various types of bargaining procedures are allowed.[6] An agent with n, or an agent with np amount of money, chooses an action in $A_n = \{a_{n1}, \ldots, a_{ns_n}\}$. Let $A = \Pi_{n=0}^{N} A_n$. For example, an action consists of an offer price and a reservation price. In this section, we confine our attention to the stationary equilibrium in which all agents choose pure strategies. As for mixed strategy equilibria, see Section 4. Let $S = \sum_{n=0}^{N} s_n$. Given an equilibrium action profile $a = (a_0, \ldots, a_N)$, where a_n is the action taken at np in the equilibrium, define $\alpha(a) = \{(n, j) \mid a_n = a_{nj}\}$.

The monetary transition resulted from transaction among a matched pair is described by a function f. When an agent with money holdings np and action a_{nj} meets an agent with $n'p$ and $a_{n'j'}$, the former's and the latter's states, i.e., money holdings, will be $n + f(n, j; n', j')$ and $n' - f(n, j; n', j')$, respectively. That is f maps an ordered pair $(n, j; n', j')$ to a non-negative integer $f(n, j; n', j')$. Here "ordered" means, for ex-

[6] See Remark 1 for the details.

ample, that the former is a seller and the latter is a buyer. When N is exogenously determined, we assume

$$N \geq n + f(n, j; n', j') \quad \text{and} \quad n' - f(n, j; n', j') \geq 0.$$

When N is endogenously determined, we assume the latter condition while the former one should be satisfied on the equilibrium path.

Next, we introduce government agents. They are programmed to follow a rule which prescribes them how to collect tax from or give subsidy to the agents they are matched with. We assume that government agents can observe current money holdings of agents they are matched with. Let $G > 0$ be the measure of the government agents. Thus the total measure of agents is $1 + G$. Note that in the following arguments G can be any small positive number.

Then we describe government's policy by $(t_0, t_1, \ldots, t_N)$, where $t_n \in [-1, 1]$, $t_0 \geq 0$, and $t_N \leq 0$. Each government agent gives subsidy p to the matched agent with n with probability $|t_n|$ when $t_n > 0$, while she collects tax p with probability $|t_n|$ when $t_n < 0$. As seen in the previous section, the budget of the government may not be balanced out of equilibria.

Let $\theta \in \mathbb{R}^L$ be the parameters of the model besides t. Of course, θ includes k, μ, and G.

We adopt Bellman equation approach. Let V_n be the value of state n, $n = 0, \ldots, N$. The variables in the model are denoted by $x = (h, V, a)$. Let $W_{nj}(x; \theta, t)$ be the value of action j at state n. Thus, in equilibria, $W_{nj}(x; \theta, t) = V_n$ holds for $(n, j) \in \alpha(a)$. Note that $W_{nj}(x; \theta, t)$ includes the utility and/or the production cost of perishable goods.

3.1 Stationary Equilibria without Tax-Subsidy

First, we present the results in the case that $t_n = 0$ for all n.

We define

$$h_{nj} = \begin{cases} h_n & \text{if } a_{nj} = a_n, \\ 0 & \text{if } a_{nj} \neq a_n. \end{cases}$$

Then by the random matching assumption and the definition of f, the inflow I_n into state n and the outflow O_n from state n are defined as follows:

$$I_n(h, a; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{(i,j,i',j')\in X_n} h_{ij}h_{i'j'} + \sum_{(i,j,i',j')\in X'_n} h_{ij}h_{i'j'}\right],$$

$$O_n(h, a; \theta) = \frac{\mu\kappa}{1+G}\left[\sum_{(j,i',j')\in Y_n} h_{nj}h_{i'j'} + \sum_{(j,i',j')\in Y'_n} h_{nj}h_{i'j'}\right],$$

where

$$\begin{aligned}
X_n &= \{(i, j, i', j') \mid f(i, j; i', j') > 0,\ i + f(i, j; i', j') = n\},\\
X'_n &= \{(i, j, i', j') \mid f(i, j, i', j') > 0,\ i' - f(i, j; i', j') = n\},\\
Y_n &= \{(j, i', j') \mid f(n, j; i', j') > 0\},\\
Y'_n &= \{(j, i', j') \mid f(i', j'; n, j) > 0\}.
\end{aligned}$$

We denote $I_n - O_n$ by D_n. Then the condition for stationarity is $D_n = 0$ for $n = 0, \ldots, N$ and $\sum_{n=0}^{N} h_n = 1$. Clearly, $\sum_{n=0}^{N} D_n = 0$ holds as an identity, and thus at least one equation is redundant. The following theorem shows that one more equation is always redundant.

Theorem 3 (Kamiya and Shimizu [6]) For any a,

$$\sum_{n=0}^{N} nD_n(h, a; \theta) = 0, \tag{14}$$

is an identity.

Suppose that two agents, say a buyer and a seller, meet and a monetary trade occurs. Then the amount of money the buyer pays is equal to that of the seller obtains; in other words, the sum of their money holdings before trade is equal to that after trade. Since this holds in each trade, the total amount of money before trades, expressed by $\sum_{n=0}^{N} pnO_n(h, a; \theta)$, is equal to the total amount of money after trades, expressed by $\sum_{n=0}^{N} pnI_n(h, a; \theta)$, and thus $\sum_{n=0}^{N} nD_n(h, a; \theta) = 0$ always holds.

Together with the other identity $\sum_{n=0}^{N} D_n(h, a; \theta) = 0$, the above theorem implies that h is a stationary distribution if and only if $D_n(h, a; \theta) = 0, n = 2, \ldots, N$, and $\sum_{n=0}^{N} h_n = 1$ hold. Namely, the condition for stationarity has at least one-degree of freedom. This is the main cause of the indeterminacy.

Now the equilibrium condition is expressed as follows:

Definition 1 Given θ, $x = (h, V, a) \in \mathbb{R}^{N+1} \times \mathbb{R}_+^{N+1} \times A$ is a (pure strategy) *stationary equilibrium without tax-subsidy* if it satisfies the following:

$$\begin{aligned} D_n(h, a; \theta) &= 0, & n &= 2, \dots, N \\ \sum_{n=0}^{N} h_n - 1 &= 0, & & \\ V_n - W_{nj}(x; \theta, 0) &= 0, & (n, j) &\in \alpha(a) \\ V_n - W_{nj}(x; \theta, 0) &\geq 0, & (n, j) &\notin \alpha(a). \end{aligned} \tag{15}$$

(h, V) is called a stationary equilibrium for a and θ if (h, V, a) is a stationary equilibrium for θ. Let E_θ^a be the set of such (h, V)s, and $g^a : \mathbb{R}_+^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{R}^L (\ni (h, V, \theta)) \to \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^{N+1} \times \mathbb{R}^{S-N-1}$ be the LHS of the above condition.

Remark 1 In addition to the above equilibrium conditions, the following conditions are typically required to be an "equilibrium" in most of matching models with money: (i) the existence of $p > 0$ satisfying $\sum_{n=0}^{N} pnh_n = M$, (ii) the incentive not to choose an action out of our action space,[7] and (iii) the incentive to take the equilibrium strategy at state $\eta \notin \{0, p, \dots, Np\}$. However, they are not very restrictive. As for (i), it immediately follows from $h_0 \neq 1$. As for (ii) and (iii), KS presents a sufficient condition to assure that (ii) and (iii) hold, and it is satisfied in all of the matching models with divisible money known so far, such as Zhou [9]'s model, a divisible money version of Camera and Corbae [2]'s model, and a divisible money version of Trejos and Wright [8]'s model.

Let

$$C^a = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-N-1},$$

and, for $(n, j) \notin \alpha(a)$,

$$C^{a(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-N-1},$$

where the last $\{0\}$ corresponds to $V_n - W_{nj}(x; \theta, 0)$. Moreover, for $(n, j), (n', j') \notin \alpha(a)$,

$$C^{a(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+1} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{S-N-1},$$

[7] For example in Section 2, a seller may offer a price which is not an integer multiple of p.

where the last two $\{0\}$s correspond to $V_n - W_{nj}(x;\theta,0)$ and $V_n - W_{n'j'}(x;\theta,0)$, respectively. Below, it is verified that there is the indeterminacy of the stationary equilibrium under some regularity conditions.

Assumption 1 [Regularity Condition] Given a, g^a is of class C^2 and is transversal to C^a, $C^{a(n,j)}$, and $C^{a(n,j)(n',j')}$ for all $(n,j) \notin \alpha(a)$ and $(n',j') \notin \alpha(a)$.[8]

Assumption 2 [Existence Condition] Given a, there exists a C^2-manifold without boundary, $\Theta \subset \mathbb{R}^L$, such that $E^a_\theta \neq \emptyset$ holds for all $\theta \in \Theta$.

Theorem 4 (Kamiya and Shimizu [6]) For a given a, suppose the Regularity Condition and the Existence Condition are satisfied for some Θ. Then, for almost every $\theta \in \Theta$, E^a_θ is a one-dimensional manifold with boundary. Moreover, at any endpoint of the manifold, only one $V_n - W_{nj}(x;\theta,0) \geq 0$, $(n,j) \notin \alpha(a)$, is binding, and at points in the relative interior of the manifold, no inequality is binding.

KS also shows that this indeterminacy is indeed a real one; i.e., the welfare are typically not the same in a connected component of the equilibrium manifold.

3.2 Stationary Equilibria with Tax-Subsidy

In this section, we investigate the case with $t \neq (0,\ldots,0)$. In what follows, variables and functions with "tilde" denote the ones with nonzero t. The inflow at n, $\tilde{I}_n$, and the outflow at n, $\tilde{O}_n$, are defined as follows:

$$\tilde{I}_n(\tilde{h},a;\theta,t) = I_n(\tilde{h},a;\theta) + \frac{\mu G}{1+G}\left(t^+_{n-1}\tilde{h}_{n-1} + t^-_{n+1}\tilde{h}_{n+1}\right),$$
$$\tilde{O}_n(\tilde{h},a;\theta,t) = O_n(\tilde{h},a;\theta) + \frac{\mu G}{1+G}|t_n|\tilde{h}_n,$$

where $t^+_n = \max\{0,t_n\}$, $t^-_n = -\min\{0,t_n\}$, and $t_{-1} = t_{N+1} = 0$. Let $\tilde{D}_n(\tilde{h},a;\theta,t) = \tilde{I}_n(\tilde{h},a;\theta,t) - \tilde{O}_n(\tilde{h},a;\theta,t)$.

Since $\sum_{n=0}^N n\tilde{D}_n$ is not identically zero, then we define a stationary equilibrium with tax-subsidy as follows.

Definition 2 Given θ, $\tilde{x} = (\tilde{h},\tilde{V},a) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}_+ \times A$ is a (pure strategy) *stationary*

[8] This assumption implies that that $D_n = 0, n = 2,\ldots,N$, are linearly independent in stationary equilibria. See KS for indeterminacy results of the other cases.

equilibrium with tax-subsidy scheme t if it satisfies the following:

$$\begin{aligned}
\tilde{D}_n(\tilde{h}, a; \theta, t) &= 0, && n = 1, \dots, N \\
\sum_{n=0}^{N} \tilde{h}_n - 1 &= 0, && \\
\tilde{V}_n - W_{nj}(\tilde{x}; \theta, t) &= 0, && (n, j) \in \alpha(a) \\
\tilde{V}_n - W_{nj}(\tilde{x}; \theta, t) &\geq 0, && (n, j) \notin \alpha(a).
\end{aligned} \tag{16}$$

Theorem 5 Given a, consider the following system of the stationary condition:

$$(\tilde{D}_1, \dots, \tilde{D}_N, \sum_{n=0}^{N} \tilde{h}_n - 1)^T = (0, \dots, 0)^T,$$

where T denotes transpose. If the Jacobian matrix with respect to $\tilde{h}$ of the LHS of the above system is of full rank at a stationary distribution, then the stationary distribution is locally determinate. Moreover, the budget is balanced on this stationary distribution.

Proof:
The first statement follows from the inverse function theorem. As for the second statement, it is verified that the budget deficit is equal to

$$\begin{aligned}
\frac{\mu G}{1 + G} \tilde{h} \cdot t &= \sum_{n=0}^{N} n \tilde{D}_n(\tilde{h}, a; \theta, t) - \sum_{n=0}^{N} n D_n(\tilde{h}, a; \theta) \\
&= \sum_{n=0}^{N} n \tilde{D}_n(\tilde{h}, a; \theta, t),
\end{aligned}$$

where the second equality follows from Theorem 3. Note that even in the case with tax-subsidy the same logic as in Theorem 3 applies. Then $\sum_{n=0}^{N} n D_n(\tilde{h}, a; \theta)$ is equal to 0 in stationary distributions with tax-subsidy, since $\tilde{D}_n(\tilde{h}, a; \theta, t) = 0, n = 0, \dots, N$. ∎

Next, we show the existence of a locally determinate stationary equilibrium which has the following property; it is induced by a certain tax-subsidy scheme, and it exists in any given neighborhood of the stationary equilibrium which is not induced by tax-subsidy. We choose an arbitrary stationary equilibrium without tax-subsidy, denoted by $x^* = (h^*, V^*, a^*)$, which is in the relative interior of the equilibrium manifold. Thus, by Theorem 4, (15) is satisfied with strict inequalities.

First we can find the following vector:

Lemma 1 There exists an $(N+1)$-dimensional vector τ satisfying
(a) $\tau \neq (0, \dots, 0)$,
(b) $\left(\frac{\partial D_n(h^*, a^*; \theta)}{\partial h_i}\right)_{i=0,\dots,N} \cdot \tau = 0$ for $n = 2, \dots, N$,
(c) $h^* \cdot \tau = 0$.

The above lemma clearly holds, since (b) and (c) have at least one-degree of freedom.

Using this vector, we construct a tax-subsidy scheme $t = \epsilon\tau$. Here $\epsilon > 0$ is the size of the policy. For such a t to be a tax-subsidy scheme, we need the following assumption:

Assumption 3 It is also satisfied for τ in Lemma 1 that
(d) $\tau_N \leq 0$, and
(e) $\tau_0 \geq 0$.

Next, we make the following assumption.

Assumption 4 W_{nj} is C^2 with respect to ϵ for any (n, j).

If this assumption holds and ϵ is sufficiently small, then all the incentive conditions in the case with tax-subsidy is also satisfied. Thus a^* is also an equilibrium action even in the case with tax-subsidy. In other words, $(\tilde{h}, \tilde{V}, a^*)$ such that $(\tilde{h}, \tilde{V})$ is in the neighborhood of (h^*, V^*) and satisfies the following conditions is a stationary equilibrium for sufficiently small $\epsilon > 0$.

$$\tilde{D}_n(\tilde{h}, a^*; \theta, \epsilon\tau) = 0, \quad n = 1, \dots, N \tag{17}$$

$$\sum_{n=0}^{N} \tilde{h}_n - 1 = 0, \tag{18}$$

$$\tilde{V}_n - W_{nj}(\tilde{h}, \tilde{V}, a^*; \theta, \epsilon\tau) = 0, \quad (n, j) \in \alpha(a^*). \tag{19}$$

Let $\tilde{g}_\epsilon^{a^*}(\tilde{h}, \tilde{V})$ be the LHS of the above equations. Then the set of stationary equilibria is equivalent to the solution set of $\tilde{g}_\epsilon^{a^*}(\tilde{h}, \tilde{V}) = (0, \dots, 0)^T$.

Furthermore, we construct $\hat{g}_\epsilon^{a^*}$ by replacing $\tilde{D}_1$ in $\tilde{g}_\epsilon^{a^*}$ by $\tilde{h} \cdot \tau$. Then for $\epsilon > 0$, the solution set of $\tilde{g}_\epsilon^{a^*}(\tilde{h}, \tilde{V}) = (0, \dots, 0)^T$ is equivalent to the solution set of $\hat{g}_\epsilon^{a^*}(\tilde{h}, \tilde{V}) = (0, \dots, 0)^T$, since

$$\frac{\mu G \epsilon}{1 + G} \tilde{h} \cdot \tau = \sum_{n=0}^{N} n \tilde{D}_n \tag{20}$$

holds on any stationary equilibrium. More precisely, in the stationarity condition (17), we can use $\tilde{h} \cdot \tau = 0$ instead of $\tilde{D}_1 = 0$. In other words, for $\epsilon > 0$, the both conditions have the same solutions because it follows from $\tilde{D}_2 = 0, \ldots, \tilde{D}_N = 0$, and (20) that $\tilde{h} \cdot \tau = 0$ imply $\tilde{D}_1 = 0$, and vice versa. In the following lemma, we show that if the Regularity Condition is satisfied, $(h^*, \mathcal{V}^*)$ is a determinate solution to $\hat{g}^{a^*}_\epsilon(\tilde{h}, \tilde{V}) = (0, \ldots, 0)^T$ at $\epsilon = 0$. Thus $(\tilde{h}(\epsilon), \tilde{\mathcal{V}}(\epsilon))$ converges to $(h^*, \mathcal{V}^*)$ as $\epsilon \to 0$ by the implicit function theorem.

Lemma 2 Under the Regularity Condition and Assumption 3, the Jacobian matrix of $\hat{g}^{a^*}_0$ with respect to $(\tilde{h}, \tilde{V})$ is of full rank at (h^*, V^*).

Proof:
Since the Jacobian matrix has the following form

$$\begin{bmatrix} \begin{bmatrix} \tau_0 & \ldots & \tau_N \\ \frac{\partial D_2}{\partial h_0} & \ldots & \frac{\partial D_2}{\partial h_N} \\ & \ldots & \\ \frac{\partial D_N}{\partial h_0} & \ldots & \frac{\partial D_N}{\partial h_N} \\ 1 & \ldots & 1 \end{bmatrix} & 0 \\ \ldots & \begin{bmatrix} J_V(V_0 - W_{0j(0)}) \\ \ldots \\ J_V(V_N - W_{Nj(N)}) \end{bmatrix} \end{bmatrix},$$

where $j(n)$ is defined as $(n, j(n)) \in \alpha(a^*)$ and $J_V(V_n - W_{nj(n)})$ is the Jacobian matrix with respect to $V = (V_0, V_1, \ldots, V_N)$, then it suffices to show that the upper-left submatrix and the lower-right submatrix are of full rank. The Regularity condition implies that the lower-right submatrix is of full rank. As for the upper-left submatrix, condition (b) implies that the 1st row vector is independent of the 2nd, ..., and Nth row vectors. Next, conditions (a), (d), and (e) imply that the 1st row vector is independent of the last row vector. By the Regularity Condition, the 2nd , ... , and Nth row vectors are mutually independent. Finally, by the Regularity Condition, the last row vector is independent of 2nd, ... , and Nth row vectors. ∎

Thus (h^*, V^*) is a locally determinate solution to $\hat{g}^{a^*}_0 = (0, \ldots, 0)^T$. Then applying the implicit function theorem to $\hat{g}^{a^*}_\epsilon = (0, \ldots, 0)^T$ at $(\tilde{h}, \tilde{V}, \epsilon) = (h^*, V^*, 0)$, it can be clearly shown that, for all $\delta > 0$, there exist $\epsilon > 0$ and $(h^*_\epsilon, V^*_\epsilon)$ such that $(h^*_\epsilon, V^*_\epsilon)$ is the locally unique solution to $\tilde{g}^{a^*}_\epsilon = (0, \ldots, 0)^T$ and is in the δ-neighborhood of (h^*, V^*). Finally, since (h^*, V^*) is in the relative interior of the equilibrium manifold, all the incentive conditions are still satisfied for a sufficiently small ϵ. Thus we obtain the following theorem.

Theorem 6 Suppose the Regularity Condition, the Existence Condition, and Assumptions 3 and 4 hold. Then, for almost every $\theta \in \Theta$, almost every $(h^*, V^*) \in E_\theta^{a^*}$, and any δ-neighborhood of (h^*, V^*), there exists a tax-subsidy scheme such that a stationary equilibrium with tax-subsidy is locally determinate and lies in the neighborhood.

4 Mixed Strategy Equilibria

In this section we deal with mixed strategy stationary equilibria.

Let $b_{nj} \geq 0$ be the proportion of the agents choosing an action a_{nj} among the agents with n, and $b = (b_{01}, \ldots, b_{nj}, \ldots, b_{Ns_N})$. Thus $\sum_{j=1}^{s_n} b_{nj} = 1$ holds. Then an equilibrium is defined in terms of $x = (h, V, b)$.

First, we present the results in the case without tax-subsidy. Let $h_{nj} = b_{nj} h_n$. Then I_n and O_n are defined similarly as in the previous section. Then we obtain the following result similar to Theorem 3.

Theorem 7 For any b,

$$\sum_{n=0}^{N} n D_n(h, b; \theta) = 0, \tag{21}$$

is an identity.

Let $\hat{B}$ be the power set of $\{(n, j) \mid j = 1, \ldots, s_n, n = 0, \ldots, N\}$ and B be $\{\beta \in \hat{B} \mid \forall n, \exists j, (n, j) \in \beta\}$. $\beta \in B$ can be considered as a set of actions used in an equilibrium. For a given $\beta \in B$, let

$$\Omega^\beta = \{(b_{nj})_{(n,j)\in\beta} \mid b_{nj} > 0 \quad \text{for } (n, j) \in \beta\}. \tag{22}$$

Let $x^\beta = (V, h, b^\beta)$, where $b^\beta \in \Omega^\beta$. For a given $\beta \in B$, $W_{nj}^\beta(x^\beta; \theta, t)$ is defined from $W_{nj}(x; \theta, t)$ by setting $b_{n'j'} = 0$ for any $(n', j') \notin \beta$. In parallel with this, $D_n^\beta(h, b^\beta; \theta)$ is defined.

Definition 3 For a given $\beta \in B$, $x^\beta = (V, h, b^\beta) \in \mathbb{R}^{N+1} \times \mathbb{R}_+^{N+1} \times \mathbb{R}_+^{\sum_{n=0}^{N} k_n}$ is a *mixed strategy stationary equilibrium without tax-subsidy* for β and θ if it satisfies the following:

$$
\begin{aligned}
D_n^{\beta}(h, b^{\beta}; \theta) &= 0, && n = 2, \ldots, N \\
\sum_{n=0}^{N} h_n - 1 &= 0, && \\
V_n - W_{nj}^{\beta}(x^{\beta}; \theta, 0) &= 0, && (n, j) \in \beta \\
\sum_{j \in \{j' | (j', n) \in \beta\}} b_{nj} - 1 &= 0, && n = 0, \ldots, N \\
V_n - W_{nj}^{\beta}(x^{\beta}; \theta, 0) &\geq 0, && (n, j) \notin \beta.
\end{aligned}
\tag{23}
$$

Let E_θ^β be the set of such an x^β, and $g^\beta : \mathbb{R}^{N+1} \times \mathbb{R}_{++}^{N+1} \times \Omega^\beta \times \mathbb{R}^L \to \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^{\#\beta} \times \mathbb{R}^{N+1} \times \mathbb{R}^{S-\#\beta}$ be the LHS of the above equations.

Let

$$
C^\beta = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#\beta+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-\#\beta},
$$

and, for $(n, j) \notin \beta$,

$$
C^{\beta(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#\beta+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{S-\#\beta},
$$

where the last $\{0\}$ corresponds to $V_n - W_{nj}^\beta(x^\beta; \theta, 0)$. Moreover, for $(n, j), (n', j') \notin \beta$ such that $(n, j) \neq (n', j')$,

$$
C^{\beta(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#\beta+1} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{S-\#\beta},
$$

where the last two $\{0\}$s correspond to $V_n - W_{nj}^\beta(x^\beta; \theta, 0)$, and $V_n - W_{n'j'}^\beta(x^\beta; \theta, 0)$, respectively.

Assumption 5 [Regularity Condition] Given β, g^β is C^2 and is transversal to C^β, $C^{\beta(n,j)}$, and $C^{\beta(n,j)(n',j')}$ for all $(n, j) \notin \beta$ and $(n', j') \notin \beta$.

Assumption 6 [Existence Condition] Given β, there exists C^2 manifold without boundary, $\Theta \subset R^L$ such that $E_\theta^\beta \neq \emptyset$ holds for all $\theta \in \Theta$.

Theorem 8 (Kamiya and Shimizu [6]) For a given β, suppose the Regularity Condition and the Existence Condition is satisfied for Θ. Then, for almost every $\theta \in \Theta$, E_θ^β is a one-dimensional manifold with boundary. Moreover, at any endpoint of the manifold, only one $V_n - W_{nj}(x; \theta, 0) \geq 0$, $(n, j) \notin \beta$, is binding, and at points in the relative interior of the manifold, no inequality is binding.

Based on these results, we investigate the case with tax-subsidy. First, we obtain almost the same result as Theorem 5.

Theorem 9 Given β, consider the following system of the stationary condition:

$$(\tilde{D}_1^\beta, \ldots, \tilde{D}_N^\beta, \sum_{n=0}^{N} \tilde{h}_n - 1)^T = (0, \ldots, 0)^T.$$

If the Jacobian matrix with respect to $\tilde{h}$ of the LHS of the above system is of full rank at a stationary distribution, then the stationary distribution is locally determinate. Moreover, the budget is balanced on this stationary distribution.

Next, we fix an arbitrary stationary equilibrium without tax-subsidy, denoted by x^*,[9] which is in the relative interior of the equilibrium manifold.

We construct a tax-subsidy scheme such that $t = \epsilon\tau$. To do so, we need the following Lemma and Assumption.

Lemma 3 There exists an $(N+1)$-dimensional vector τ satisfying
(a) $\tau \neq (0, \ldots, 0)$,
(b) $\left(\frac{\partial D_n^{\beta^*}(h^*, b^{*\beta^*}; \theta)}{\partial h_i}\right)_{i=0,\ldots,N} \cdot \tau = 0$ for $n = 2, \ldots, N$,
(c) $h^* \cdot \tau = 0$.

Assumption 7 It is also satisfied for τ in Lemma 3 that
(d) $\tau_N \leq 0$, and
(e) $\tau_0 \geq 0$.

Also we make the following assumption.

Assumption 8 $\tilde{W}_{nj}^{\beta^*}$ is C^2 with respect to ϵ for any (n, j).

[9] To be strict, we should denote by $x^{*\beta^*}$, but we simply do by x^* to avoid a complicated notation.

As in the case of pure strategies, for a sufficiently small ϵ, we express the condition for a stationary equilibrium with tax-subsidy as follows:

$$\begin{aligned} \tilde{D}_n^{\beta^*}(\tilde{h}, \tilde{b}^{\beta^*}; \theta, \epsilon\tau) = 0, & \qquad n = 1, \ldots, N \\ \sum_{n=0}^{N} \tilde{h}_n - 1 = 0, & \\ \tilde{V}_n - W_{nj}^{\beta^*}(\tilde{x}^{\beta^*}; \theta, \epsilon\tau) = 0, & \qquad (n, j) \in \beta^*. \end{aligned}$$

Let $\tilde{g}_\epsilon^{\beta^*}(\tilde{x}^{\beta^*})$ be the LHS of the above equations. Then the set of stationary equilibria is equivalent to the solution set of $\tilde{g}_\epsilon^{\beta^*}(\tilde{x}^{\beta^*}) = (0, \ldots, 0)^T$.

Furthermore, we construct $\hat{g}_\epsilon^{\beta^*}$ by replacing $\tilde{D}_1^{\beta^*}$ in $\tilde{g}_\epsilon^{\beta^*}$ by $\tilde{h} \cdot \tau$. Then for $\epsilon > 0$, the solution set of $\tilde{g}_\epsilon^{\beta^*}(\tilde{x}^{\beta^*}) = (0, \ldots, 0)^T$ is equivalent to the solution set of $\hat{g}_\epsilon^{\beta^*}(\tilde{x}^{\beta^*}) = (0, \ldots, 0)^T$. We need to make the assumption on $\hat{g}_\epsilon^{a^*}$:

Assumption 9 The Jacobian matrix of $\hat{g}_0^{\beta^*}$ with respect to $\tilde{x}^{\beta^*}$ is of full rank at x^*.

This assumption implies x^* is a locally determinate solution to $\hat{g}_0^{\beta^*} = (0, \ldots, 0)^T$. Then applying the implicit function theorem to $\hat{g}_\epsilon^{\beta^*} = (0, \ldots, 0)^T$ at $(x^{\beta^*}, \epsilon) = (x^*, 0)$, it can be clearly shown that, for any $\delta > 0$, there exist $\epsilon > 0$ and $(x_\epsilon^*$ such that x_ϵ^* is the locally unique solution to $\tilde{g}_\epsilon^{\beta^*} = (0, \ldots, 0)^T$ and is in the δ-neighborhood of x^*. Finally, since x^* is in the relative interior of the equilibrium manifold, all the incentive conditions are still satisfied for a sufficiently small ϵ. Thus we obtain the following theorem.

Theorem 10 Suppose the Regularity Condition, the Existence Condition, and Assumptions 8 and 9 hold. Then for almost every $\theta \in \Theta$, almost every $x^* \in E_\theta^{\beta^*}$, and any δ-neighborhood of x^*, there exists a tax-subsidy scheme such that a stationary equilibrium with tax-subsidy is locally determinate and lies in the neighborhood.

5 Conclusion

In this paper, we show that although there is a continuum of stationary equilibria in money search models, some tax-subsidy schemes can select a determinate efficient one among them. In other words, we find a new role of the tax-subsidy schemes. It is notable that a small amount of tax-subsidy is enough for this role.

If the amounts of tax and subsidy are relatively large, the government may obtain a more efficient equilibrium than those without tax-subsidy. Thus it is the most important future research to seek for the best policy allowing for a tax-subsidy scheme of relatively large amounts.

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Appendix

A The Definition of Stationary Equilibrium in Zhou's Model

From h, the stationary distribution of offer prices, Ω, and the stationary distribution of reservation prices, R, are defined as follows.

$$\Omega(x) = \sum_{n \in \{n' \in \mathbb{N} | \omega(n'p) \leq x\}} h_n, \tag{24}$$

$$R(x) = \sum_{n \in \{n' \in \mathbb{N} | \rho(n'p) < x\}} h_n. \tag{25}$$

Let $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}_+$ be a value function. Then, using γ, μ, and h, the Bellman equation for $\mathcal{V}(\eta)$ is given by

$$\begin{aligned}\mathcal{V}(\eta) = \frac{1}{\phi + 2 + kG} \Bigg[& \max_{r \in [0,\eta]} \left\{ \int_0^r (u + \mathcal{V}(\eta - x))\, d\Omega(x) + (1 - \Omega(r))\, \mathcal{V}(\eta) \right\} \\ & + \max_{o \in \mathbb{R}_+ \cup \{NT\}} S(o) + (kG|t_{\lfloor \frac{\eta}{p} \rfloor}|\mathcal{V}(\eta + \text{sign}(t_{\lfloor \frac{\eta}{p} \rfloor})p) + kG(1 - |t_{\lfloor \frac{\eta}{p} \rfloor}|)\mathcal{V}(\eta)) \Bigg],\end{aligned} \tag{26}$$

where $\lfloor y \rfloor$ is the integer part of y, and

$$S(o) = \begin{cases} R(o)\mathcal{V}(\eta) + (1 - R(o)) \; (\mathcal{V}(\eta + o) - c)\,, & \text{if } o \in \mathbb{R}_+, \\ \mathcal{V}(\eta), & \text{if } o = NT. \end{cases} \tag{27}$$

The first term in the bracket of the RHS of (26) is the value when an agent is a buyer, the second term is the value when she is a seller, and the third term is the value when she meets a government agent. If $\rho(\eta)$ and $\omega(\eta)$ are the maximizers of the above equation, it can be rewritten as

$$\begin{aligned}\mathcal{V}(\eta) = \frac{1}{\phi + 2 + kG} \Bigg[& \int_0^{\rho(\eta)} (u + \mathcal{V}(\eta - x))\, d\Omega(x) + (1 - \Omega(\rho(\eta)))\, \mathcal{V}(\eta) + S(\omega(\eta)) \\ & + (kG|t_{\lfloor \frac{\eta}{p} \rfloor}|\mathcal{V}(\eta + \text{sign}(t_{\lfloor \frac{\eta}{p} \rfloor})p) + kG(1 - |t_{\lfloor \frac{\eta}{p} \rfloor}|)\mathcal{V}(\eta)) \Bigg].\end{aligned} \tag{28}$$

In terms of $\mathcal{V}(\eta)$, it is optimal to accept offer $o \in \mathbb{R}_+$ if $u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)$. That is the optimal offer strategy ρ satisfies $\rho(\eta) \geq o$ if and only if $u + \mathcal{V}(\eta - o) \geq \mathcal{V}(\eta)$. For the perfectness of equilibria, this should hold even in off-equilibrium-paths. Then, in case that a value function is continuous from the right, the perfectness condition with respect to reservation price is as follows:

$$\rho(\eta) = \max\left\{ r \in [0, \eta] \middle| u + \mathcal{V}(\eta - r) \geq \mathcal{V}(\eta) \right\}. \tag{29}$$

That is, type i's reservation price is her full value for good $i+1$, and thus it is a function of η. In order to assure that (29) is actually defined, we confine our attention to the case that a value function is continuous from the right.

The economy is stationary if h is an initial stationary distribution of the process induced by the optimal trading strategy (ω, ρ). Now we define the stationary equilibrium grounded on the above.

Definition 4 $\langle h, \omega, \rho, \mathcal{V} \rangle$, where $\mathcal{V}$ is a step function with step $p > 0$, is said to be a *stationary equilibrium* if

1. h is stationary under trading strategies ω and ρ, and the distribution of offer prices Ω and that of reservation prices R are derived from (24) and (25),
2. $\sum_{n=0}^{N} pnh_n = M$, and
3. given the distributions h, R and Ω, the reservation price strategy ρ satisfies the feasibility condition (1) and the perfectness condition (29), respectively, and the value function $\mathcal{V}$, together with ρ and ω, solves the Bellman equation (26).

Remark 2 In the case without tax-subsidy, i.e., $t = (0, \ldots, 0)$, (28) can be rewritten as

$$\mathcal{V}(\eta) = \frac{1}{\phi + 2} \left[\int_0^{\rho(\eta)} \{u + \mathcal{V}(\eta - x)\} \, d\Omega(x) + \{1 - \Omega(\rho(\eta))\} \mathcal{V}(\eta) + S(\omega(\eta)) \right].$$

This is essentially the same as the value function in Zhou [9], though the definition of ϕ is slightly different. Thus all the results in Zhou [9] and Kamiya et al. [5] hold even in our model with $t = (0, \ldots, 0)$.

B The Regularity of Single Price Equilibria

Let $V_n = \mathcal{V}(np), n = 0, 1, \ldots$. Then $\tilde{h} = (\tilde{h}_0, \ldots, \tilde{h}_N)$ and $\tilde{V} = (\tilde{V}_0, \ldots, \tilde{V}_N)$ should satisfy the following equations in stationary equilibria:

$$F_0 = \tilde{h}_0 + \cdots + \tilde{h}_N - 1 = 0, \tag{30}$$

$$F_1 = \tilde{h} \cdot \tau = 0, \tag{31}$$

$$\begin{aligned} F_n = {} & \tilde{h}_{n-1}(1 - \tilde{h}_0) + \tilde{h}_{n+1}(1 - \tilde{h}_N) + kG\epsilon(t^+_{n-1}\tilde{h}_{n-1} + t^-_{n+1}\tilde{h}_{n+1}) \\ & - \tilde{h}_n(1 - \tilde{h}_0 + 1 - \tilde{h}_N + kG\epsilon|\tau_n|) = 0, \quad n = 2, \ldots, N-1, \end{aligned} \tag{32}$$

$$F_N = \tilde{h}_{N-1}(1 - \tilde{h}_0) + kG\epsilon\tau^+_{N-1}\tilde{h}_{N-1} - \tilde{h}_N(1 - \tilde{h}_N) + kG\epsilon\tau_N\tilde{h}_N = 0, \tag{33}$$

$$\begin{aligned} G_0 = {} & \tilde{V}_0 - \frac{1}{\phi + 2 + kG}\Big\{(1 - \tilde{h}_0)(\tilde{V}_1 - c) + \tilde{h}_0\tilde{V}_0 + \tilde{V}_0 \\ & + kG\epsilon\tau_0\tilde{V}_1 + kG(1 - \epsilon\tau_0)\tilde{V}_0\Big\} = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} G_n = {} & \tilde{V}_n - \frac{1}{\phi + 2 + kG}\Big\{(1 - \tilde{h}_0)(\tilde{V}_{n+1} - c) + \tilde{h}_0\tilde{V}_n + (1 - \tilde{h}_N)(u + \tilde{V}_{n-1}) + \tilde{h}_N\tilde{V}_n \\ & + kG\epsilon\tau^+_n\tilde{V}_{n+1} + kG\epsilon\tau^-_n\tilde{V}_{n-1} + kG(1 - \epsilon|\tau_n|)\tilde{V}_n\Big\} = 0, \quad n = 1, \ldots, N-1, \end{aligned} \tag{35}$$

$$\begin{aligned} G_N = {} & \tilde{V}_N - \frac{1}{\phi + 2 + kG}\Big\{\tilde{V}_N + (1 - \tilde{h}_N)(u + \tilde{V}_{N-1}) + \tilde{h}_N\tilde{V}_N \\ & - kG\epsilon\tau_N\tilde{V}_{N-1} + kG(1 + \epsilon\tau_N)\tilde{V}_N\Big\} = 0, \end{aligned} \tag{36}$$

where $t^+_n = \max\{0, t_n\}$ and $t^-_n = -\min\{0, t_n\}$. (30) simply says that the total measure is one. (31) is the equation introduced instead of the stationarity condition at $n = 1$. (32) and (33) are the conditions for stationary of money holdings distribution. The last three equations (34)-(36) are the conditions that the specified strategy indeed realizes the value.

Next, let $\Psi : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \times \mathbb{R} \times \mathbb{R}^{N+1} \to \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$ be defined as

$$\Psi\left(\tilde{h}, \tilde{V}, \epsilon, \tau\right) = (F_0, \ldots, F_N, G_0, \ldots, G_N)(\tilde{h}, \tilde{V}, \epsilon, \tau).$$

Recall that (h^*, V^*) is a SPE without tax-subsidy. Clearly, $\Psi(h^*, V^*, 0, \tau) = 0$ holds for any τ. Note that Ψ is C^2. For a given τ, if the determinant of Ψ w.r.t. $(\tilde{h}, \tilde{V})$ at $(h^*, V^*, 0, \tau)$ is not zero, then by the implicit function theorem, $\Psi(\tilde{h}, \tilde{V}, \epsilon, \tau) = 0$ can be solved for $(\tilde{h}, \tilde{V})$ in terms of ϵ, in a small neighborhood of (h^*, V^*), and these functions are continuously differentiable functions of ϵ.

We restrict our attention to the set of $\tau_1, \ldots, \tau_N$ such that $\sum_{n=1}^N \tau_n h^*_n < 0$ and h such that $h_n > 0, n = 0, \ldots, N$. Let $\tau_0 = -\frac{\sum_{n=1}^N \tau_n h^*_n}{h^*_0}$. The Jacobian matrix of Ψ w.r.t.

$\tilde{h}, \tilde{V}$, and $\tau_1, \ldots, \tau_N$ is as follows:

$$A = \begin{bmatrix} \Upsilon_1 & 0 & \Upsilon_3 \\ \Upsilon_2 & \Upsilon_4 & \Upsilon_5 \end{bmatrix},$$

where Υ_1 and Υ_2 are the Jacobian matrices of $(F_0, \ldots, F_N)$ and $(G_0, \ldots, G_N)$ w.r.t. $\tilde{h}$, respectively, Υ_4 is the Jacobian matrix $(G_0, \ldots, G_N)$ w.r.t. $\tilde{V}$, and Υ_3 and Υ_5 are the Jacobian matrices of $(F_0, \ldots, F_N)$ and $(G_0, \ldots, G_N)$ w.r.t. $\tau_1, \ldots, \tau_N$, respectively.

First, we consider the case of $(h_0^*, \ldots, h_N^*) \neq \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$. Suppose $\sum_{n=0}^{N} \tau_n \neq 0$. Then, for any $(\tilde{h}, \tilde{V}, \tau_1, \ldots, \tau_N)$ satisfying $\Psi\left(\tilde{h}, \tilde{V}, 0, \tau\right) = 0$, $\tilde{h}_N \neq \frac{1}{N+1}$ holds, since $\tilde{h} = \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right)$ is the unique stationary distribution satisfying $\tilde{h}_N = \frac{1}{N+1}$, (30), (32) and (33), and this does not satisfy (31), i.e.,

$$(\tau_0, \ldots, \tau_N) \cdot \left(\frac{1}{N+1}, \ldots, \frac{1}{N+1}\right) = \frac{1}{N+1} \sum_{n=0}^{N} \tau_n \neq 0.$$

Note that if $\tilde{h}_N \neq \frac{1}{N+1}$ holds, then $\tilde{h}_N \neq \tilde{h}_0$ holds in stationary distributions satisfying (30), (32) and (33).

Then we will show that A is always of full rank at equilibria in the set of $(\tau_1, \ldots, \tau_N)$ such that $\sum_{n=0}^{N} \tau_n \neq 0$, and thus by the parametric transversality theorem the Jacobian matrix of Ψ w.r.t. $\tilde{h}$ and $\tilde{V}$ is of full rank at an equilibrium for almost every $\tau_1, \ldots, \tau_N$ in the set. To see this, we show that, Υ_4 and the matrix consists of the first column of Υ_3 and the second to the last column of Υ_1, denoted by Υ_6, are of full rank.

Since, at $\epsilon = 0$,

$$\Upsilon_1 = \begin{bmatrix} 1 & \cdots & & & & & & \cdots & 1 \\ \tau_0 & \cdots & & & & & & \cdots & \tau_N \\ -\tilde{h}_1 + \tilde{h}_2 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & 0 & \cdots & \cdots & 0 & \tilde{h}_2 - \tilde{h}_3 \\ \vdots & & & & & & & & \\ -\tilde{h}_{N-3} + \tilde{h}_{N-2} & 0 & \cdots & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & \tilde{h}_{N-2} - \tilde{h}_{N-1} \\ -\tilde{h}_{N-2} + \tilde{h}_{N-1} & 0 & \cdots & & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 + \tilde{h}_{N-1} - 2\tilde{h}_N \\ -\tilde{h}_{N-1} & 0 & \cdots & & & \cdots & 0 & 1 - \tilde{h}_0 & -1 + 2\tilde{h}_N \end{bmatrix}$$

and

$$\Upsilon_3 = \begin{bmatrix} 0 & \cdots & 0 \\ \tilde{h}_1 & \cdots & \tilde{h}_N \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix},$$

then Υ_6 at $\epsilon = 0$ is expressed as follows:

$$\Upsilon_6 = \begin{bmatrix} 0 & 1 & & & & & & \cdots & 1 \\ \tilde{h}_1 & \tau_1 & & & & & & \cdots & \tau_N \\ 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & \cdots & & \cdots & 0 & \tilde{h}_2 - \tilde{h}_3 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & \tilde{h}_{N-2} - \tilde{h}_{N-1} \\ 0 & 0 & \cdots & & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 + \tilde{h}_{N-1} - 2\tilde{h}_N \\ 0 & 0 & \cdots & & & \cdots & 0 & 1 - \tilde{h}_0 & -1 + 2\tilde{h}_N \end{bmatrix}$$

Thus

$$|\Upsilon_6| = \frac{1}{\tilde{h}_N - \tilde{h}_0} \begin{vmatrix} 0 & -1+\tilde{h}_N & 0 & \dots & & & & \dots & 0 & \tilde{h}_1 \\ \tilde{h}_1 & \tau_1 & & & & & & & \dots & \tau_N \\ 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \dots & & & \dots & 0 & \tilde{h}_2 \\ 0 & 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \dots & & \dots & 0 & \tilde{h}_3 \\ \vdots & & & & & & & & & \\ 0 & 0 & \dots & & \dots & 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \tilde{h}_{N-2} \\ 0 & 0 & \dots & & & \dots & 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & \tilde{h}_{N-1} \\ 0 & 0 & \dots & & & & \dots & 0 & 1-\tilde{h}_0 & -1+2\tilde{h}_N \end{vmatrix}$$

$$= \frac{\tilde{h}_1}{\tilde{h}_0 - \tilde{h}_N} \begin{vmatrix} -1+\tilde{h}_N & 0 & \dots & & & & \dots & 0 & \tilde{h}_1 \\ 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \dots & & & \dots & 0 & \tilde{h}_2 \\ 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \dots & & \dots & 0 & \tilde{h}_3 \\ \vdots & & & & & & & & \\ 0 & \dots & & \dots & 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & 0 & \tilde{h}_{N-2} \\ 0 & \dots & & & \dots & 0 & 1-\tilde{h}_0 & -1+\tilde{h}_N & \tilde{h}_{N-1} \\ 0 & \dots & & & & \dots & 0 & 1-\tilde{h}_0 & -1+2\tilde{h}_N \end{vmatrix}$$

$$= \frac{\tilde{h}_1}{\tilde{h}_0 - \tilde{h}_N} \left[\sum_{n=1}^{N} (-1)^{N+n} \tilde{h}_n \left(1-\tilde{h}_0\right)^{N-n} \left(-1+\tilde{h}_N\right)^{n-1} + \left(-1+\tilde{h}_N\right)^N \right].$$

Then substituting $\tilde{h}_n = \left(\frac{1-\tilde{h}_0}{1-\tilde{h}_N}\right)^n \tilde{h}_0$, $|\Upsilon_6|$ at $\epsilon = 0$ is expressed as follows:

$$\begin{aligned} |\Upsilon_6| &= \frac{\tilde{h}_1(-1+\tilde{h}_N)^{N-1}}{\tilde{h}_0 - \tilde{h}_N} \left[N \left(\frac{1-\tilde{h}_0}{1-\tilde{h}_N} \right)^N \tilde{h}_0 - (1-\tilde{h}_N) \right] \\ &= \frac{\tilde{h}_1(-1+\tilde{h}_N)^{N-1}}{\tilde{h}_0 - \tilde{h}_N} \left[N\tilde{h}_N - (1-\tilde{h}_N) \right] \\ &= \frac{\tilde{h}_1(-1+\tilde{h}_N)^{N-1}}{\tilde{h}_0 - \tilde{h}_N} \left[(N+1)\tilde{h}_N - 1 \right] \\ &\neq 0. \end{aligned}$$

Thus by the parametric transversality theorem $|\Upsilon_1| \neq 0$ holds in stationary distributions for almost every $(\tau_1, \dots, \tau_N)$ in the space.

On the other hand, at $\epsilon = 0$,

$$|\Upsilon_4| = \frac{1}{(\phi+2+kG)^{N+1}} \begin{vmatrix} \phi+1-\tilde{h}_0 & -1+\tilde{h}_0 & 0 & \dots & & & \dots & 0 \\ -1+\tilde{h}_N & \phi+2-\tilde{h}_0-\tilde{h}_N & -1+\tilde{h}_0 & 0 & \dots & & \dots & 0 \\ \vdots & & & & & & & \\ 0 & \dots & & \dots & 0 & -1+\tilde{h}_N & \phi+2-\tilde{h}_0-\tilde{h}_N & -1+\tilde{h}_0 \\ 0 & \dots & & & \dots & 0 & -1+\tilde{h}_N & \phi+1-\tilde{h}_N \end{vmatrix}$$

$$= \frac{1}{(\phi+2+kG)^{N+1}(N+1)^{N+1}} \begin{vmatrix} (N+1)\phi+N & -N & 0 & \dots & & & \dots & 0 \\ -N & (N+1)\phi+2N & -N & 0 & \dots & & \dots & 0 \\ \vdots & & & & & & & \\ 0 & \dots & & \dots & 0 & -N & (N+1)\phi+2N & -N \\ 0 & \dots & & & \dots & 0 & -N & (N+1)\phi+N \end{vmatrix}$$

Then defining $(\upsilon_n)_{n=0,\ldots,N}$ as

$$
\begin{aligned}
\upsilon_0 &= (N+1)\phi + N, \\
\upsilon_n &= (N+1)\phi + 2N - \frac{N^2}{\upsilon_{n-1}}, \quad n = 1, \ldots, N-1, \\
\upsilon_N &= (N+1)\phi + N - \frac{N^2}{\upsilon_{N-1}},
\end{aligned}
$$

we obtain

$$
|\Upsilon_4| = \frac{\prod_{n=0}^{N} \upsilon_n}{(\phi + 2 + kG)^{N+1}(N+1)^{N+1}}.
$$

Since we can show that

$$
\begin{aligned}
&\upsilon_0 > N, \\
&\upsilon_{n-1} > N \Rightarrow \upsilon_n > N,
\end{aligned}
$$

for $n = 1, \ldots, N-1$, and

$$
\upsilon_{N-1} > N \Rightarrow \upsilon_N > 0,
$$

then $|\Upsilon_4| > 0$ holds at $\epsilon = 0$.

Next, we consider the case of $h_0^* = \cdots = h_N^* = \frac{1}{N+1}$. In this case, for any $(\tilde{h}, \tilde{V}, \tau_1, \ldots, \tau_n)$ such that $\Psi\left(\tilde{h}, \tilde{V}, 0, \tau\right) = 0$, $\tilde{h}_0 = \cdots = \tilde{h}_N = \frac{1}{N+1}$ holds. Therefore we can directly show that Υ_4 and Υ_1 are of full rank for almost every $\tau_1, \ldots, \tau_N$.

$|\Upsilon_1|$ at $\epsilon = 0$ is expressed as follows:

$$
\begin{aligned}
|\Upsilon_1| &= \begin{vmatrix}
1 & \cdots & & & & & & & \cdots & 1 \\
\tau_0 & \cdots & & & & & & & \cdots & \tau_N \\
-\tilde{h}_1 + \tilde{h}_2 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & 0 & \cdots & & \cdots & 0 & \tilde{h}_2 - \tilde{h}_3 \\
\vdots & & & & & & & & & \\
-\tilde{h}_{N-3} + \tilde{h}_{N-2} & 0 & \cdots & & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 - \tilde{h}_N & \tilde{h}_{N-2} - \tilde{h}_{N-1} \\
-\tilde{h}_{N-2} + \tilde{h}_{N-1} & 0 & \cdots & & & \cdots & 0 & 1 - \tilde{h}_0 & -2 + \tilde{h}_0 + \tilde{h}_N & 1 + \tilde{h}_{N-1} - 2\tilde{h}_N \\
-\tilde{h}_{N-1} & 0 & \cdots & & & & \cdots & 0 & 1 - \tilde{h}_0 & -1 + 2\tilde{h}_N
\end{vmatrix} \\
&= \frac{1}{(N+1)^{N-1}} \begin{vmatrix}
1 & \cdots & & & & & & & \cdots & 1 \\
\tau_0 & \cdots & & & & & & & \cdots & \tau_N \\
0 & N & -2N & N & 0 & \cdots & & & \cdots & 0 \\
\vdots & & & & & & & & & \\
0 & \cdots & & & \cdots & 0 & N & -2N & N & 0 \\
0 & \cdots & & & & \cdots & 0 & N & -2N & N \\
-1 & 0 & \cdots & & & & \cdots & 0 & N & -N+1
\end{vmatrix} \\
&= \frac{1}{(N+1)^{N-1}} \begin{vmatrix}
N+1 & -\frac{N(N+1)}{2} & 2 & \cdots & & & \cdots & N-1 & N \\
0 & -\sum_{n=1}^{N} n\tau_n & \sum_{n=1}^{2} \tau_n & \cdots & & & \cdots & \sum_{n=1}^{N-1} \tau_n & \sum_{n=1}^{N} \tau_n \\
0 & 0 & -N & 0 & \cdots & & & \cdots & 0 \\
0 & 0 & 0 & -N & 0 & \cdots & & \cdots & 0 \\
\vdots & & & & & & & & \\
0 & \cdots & & & & \cdots & 0 & -N & 0 \\
0 & \cdots & & & & & \cdots & 0 & 1
\end{vmatrix} \\
&= -\left(\frac{-N}{N+1}\right)^{N-2} \sum_{n=1}^{N} n\tau_n.
\end{aligned}
$$

Then choosing any τ satisfying $\sum_{n=1}^{N} n\tau_n \neq 0$ and $\sum_{n=0}^{N} \tau_n = 0$, $|\Upsilon_1|$ is nonzero at $\epsilon = 0$. In other words, for almost every $\tau_1, \dots, \tau_N$, $|\Upsilon_1| \neq 0$ at $\epsilon = 0$. Moreover, as shown in the previous case, $|\Upsilon_4| > 0$ holds at $\epsilon = 0$.

Therefore the Jacobian matrix of Ψ w.r.t. $(\tilde{h}, \tilde{V})$ is nonzero. Then, by the implicit function theorem, $V_n(\epsilon), n = 0, \dots, N$, and $\tilde{h}(\epsilon), n = 0, \dots, N$ are C^1 functions of ϵ.

Finally, we recursively define

$$\tilde{V}_n(\epsilon) = \frac{1}{\phi + 2 + kG}\left\{\tilde{V}_n(\epsilon) + (1 - \tilde{h}_N(\epsilon))(u + \tilde{V}_{n-1}(\epsilon)) + \tilde{h}_n(\epsilon)\tilde{V}_n(\epsilon)\right\}, n = N+1, N+2, \dots,$$

and $\tilde{\mathcal{V}}\ (\eta)(\epsilon) = \tilde{V}_{\lfloor \frac{\eta}{p} \rfloor}(\epsilon)$. Note that strict incentive conditions at $n = 0, 1, \dots, N$ imply those at all $\eta \in \mathbb{R}_+$.

C SPEs with $N = 1$

We first consider the case without tax-subsidy. The stationarity condition for $h = (h_0, h_1)$ is expressed as

$$\begin{aligned}
&\frac{\mu\kappa}{1+G}\left[h_1(1-h_1) - h_0(1-h_0)\right] = 0,\\
&\frac{\mu\kappa}{1+G}\left[h_0(1-h_0) - h_1(1-h_1)\right] = 0,\\
&h_0 + h_1 = 1.
\end{aligned}$$

However, for any $h_0 \in (0,1)$, $h = (h_0, 1 - h_0)$ satisfies the stationarity condition. Also for any $h_0 \in (0,1)$, p is determined as follows:

$$\frac{M}{p} = 1 - h_0.$$

Next, we consider the values at $\{0, p, \dots\}$. We denote the value at np by V_n, then we obtain

$$\begin{aligned}
V_0 &= \frac{1}{\phi + 2 + kG}[(1-h_0)(-c + V_1) + h_0 V_0 + V_0 + kGV_0],\\
V_n &= \frac{1}{\phi + 2 + kG}[V_n + (1-h_1)(u + V_{n-1}) + h_1 V_n + kGV_n], \quad n \geq 1.
\end{aligned}$$

Solving this system of equation, we obtain

$$V_n = \frac{1}{\phi}\left[h_0 u - \left(\frac{h_0}{\phi + h_0}\right)^n \frac{\phi + h_0}{\phi + 1}\left\{h_0 u + (1 - h_0)c\right\}\right], \quad n \geq 0.$$

Let

$$\mathcal{V}(\eta) = V(\lfloor \eta/p \rfloor).$$

Then all we have to do is to check the incentive conditions.

The incentive conditions with strict inequalities are as follows:

$$\begin{aligned} &-c + V_1 > V_0, \\ &V_n > -c + V_{n+1}, \quad n \geq 1, \\ &u + V_{n-1} > V_n, \quad n \geq 1. \end{aligned}$$

The first inequality is the condition that an agent with no money has incentive to sell her production good. The second inequality is the condition that an agent with np does not have incentive to sell her production good. The third inequality is the condition that an agent with np has incentive to accept an offer price p. Note that the conditions at the other η follow from the above condition. (See Zhou [9].) The necessary and sufficient condition for the above inequalities is as follows:[10]

$$\frac{\phi}{h_0} + 1 < \frac{u}{c} < \frac{\phi(\phi + 1 + h_0)}{(h_0)^2} + 1. \tag{37}$$

In other words, if

$$\phi + 1 < \frac{u}{c} < (\phi + 1)^2, \tag{38}$$

holds, then, for any $h_0 \in \left(\frac{\phi}{(u/c)-1}, 1\right)$, the corresponding h and V constitute a stationary equilibrium in which all the relevant incentive conditions are satisfied with strict inequalities.

On the other hand, the welfare is expressed as

$$W = \frac{h_0(1 - h_0)}{\phi}(u - c).$$

It is easily seen that W has a single peak at $h = (1/2, 1/2)$ with $W = \frac{u-c}{4\phi}$.

Now consider the case with tax-subsidy. We consider the tax-subsidy scheme $t = \epsilon\tau$. Then the stationarity condition for $\tilde{h}$ is

$$\begin{aligned} &\frac{\mu\kappa}{1+G}\left[\tilde{h}_1(\tilde{h}_0 - kG\epsilon\tau_1) - \tilde{h}_0(\tilde{h}_1 + kG\epsilon\tau_0)\right] = 0, \\ &\frac{\mu\kappa}{1+G}\left[\tilde{h}_0(\tilde{h}_1 + kG\epsilon\tau_0) - \tilde{h}_1(\tilde{h}_0 - kG\epsilon\tau_1)\right] = 0, \\ &\tilde{h}_0 + \tilde{h}_1 = 1. \end{aligned}$$

[10]The following condition is slightly different from the one obtained in Zhou [9], since she adopts a different equilibrium concept.

Then for any $\epsilon > 0$, we obtain the unique solution

$$(\tilde{h}_0, \tilde{h}_1) = \left(\frac{-\tau_1}{\tau_0 - \tau_1}, \frac{\tau_0}{\tau_0 - \tau_1} \right).$$

Note that $\tilde{h}$ is orthogonal to τ. Setting $\tau_0 = 1$ and $\tau_1 = -1$, $(\tilde{h}_0, \tilde{h}_1) = (\frac{1}{2}, \frac{1}{2})$ is obtained and the corresponding (V_0, V_1) is close to optimal.

Next, we obtain the values at $\{0, p, \dots\}$ as follows:

$$\begin{aligned}
V_0 &= \frac{1}{\phi + 2 + kG}[(1 - \tilde{h}_0)(-c + V_1) + \tilde{h}_0 V_0 + V_0 + kGt_0 V_1 + kG(1 - t_0)V_0], \\
V_1 &= \frac{1}{\phi + 2 + kG}[V_1 + (1 - \tilde{h}_1)(u + V_0) + \tilde{h}_1 V_1 - kGt_1 V_0 + kG(1 + t_1)V_1], \\
V_n &= \frac{1}{\phi + 2 + kG}[V_n + (1 - \tilde{h}_1)(u + V_{n-1}) + \tilde{h}_1 V_n + kGV_n], \quad n \geq 1.
\end{aligned}$$

Solving this system of equations, we obtain

$$\begin{aligned}
V_0 &= \frac{\tilde{h}_0(1 - \tilde{h}_0 + kGt_0)u - (1 - \tilde{h}_0)(\phi + \tilde{h}_0 - kGt_1)c}{\phi\{\phi + 1 + kG(t_0 - t_1)\}}, \\
V_n &= \frac{1}{\phi}\left[\tilde{h}_0 u - \left(\frac{\tilde{h}_0}{\phi + \tilde{h}_0}\right)^{n-1} \frac{\tilde{h}_0 - kGt_1}{\phi + 1 + kG(t_0 - t_1)} \left\{\tilde{h}_0 u + (1 - \tilde{h}_0)c\right\}\right], \quad n \geq 1.
\end{aligned}$$

Then, since any V_n is continuous in ϵ, all the incentive conditions are satisfied whenever $\tilde{h}$ satisfies (37) and ϵ is sufficiently small.

Suppose (38) holds. Choose any $h_0^* \in \left(\frac{\phi}{(u/c)-1}, 1\right)$ and set $\tau^* = (1 - h_0^*, -h_0^*)$. Then set

$$\epsilon^* = \begin{cases} \frac{1}{kG} h_0^* \left(\frac{u}{c} - 1\right) - \phi, & \text{if } h_0^* \leq \sqrt{\frac{\phi}{\frac{u}{c}-1}}, \\ \frac{1}{kG} \min\left\{ h_0^* \left(\frac{u}{c} - 1\right) - \phi, \frac{\phi(\phi + h_0^*)}{(h_0^*)^2 \frac{u}{c} - (\phi + (h_0^*)^2)} - 1 \right\}, & \text{otherwise.} \end{cases}$$

Then it is verified that $\epsilon^* > 0$ and that, for any $\epsilon \in (0, \epsilon^*)$, $(h_0^*, 1 - h_0^*)$ is the distribution in the unique SPE with $N = 1$.