A Simple Example of Instability and a New Condition for Stability in Matching with Couples.*

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Abstract

It is known that the matching models with couples do not always have stable outcomes. The existing literature demonstrates this fact by means of examples. In those examples, however, one need to check a large number of outcomes and it is not easy to see what aspects of the examples are crucial for the non-existence of a stable outcome. The present paper provides a much clearer explanation with a simple example. This example also reveals that weak responsiveness, defined by Klaus and Klijn (2004) as a condition of couple's preferences, is not sufficient for the existence of a stable outcome. We show that a new condition, "reasonable responsiveness", is a sufficient condition, and it defines the maximal domain for the existence.
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1 Introduction

Some entry-level labor markets have clearing houses that employ explicit algorithms to match workers with firms. A prime example is the market for American doctors (National Resident Matching Program, or NRMP), which uses well-known algorithm that always produces *stable* outcomes in a simple

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environment.[1] A stable outcome is a desirable allocation of workers that is free from justifiable objections. Formally, an outcome is stable if (a) each firm and worker has an acceptable match and (b) no firm and worker prefer one another to their current matches.

Today, the growing number of working couples who participate in this type of matching program (e.g. NRMP) invites some problems. Typically, a working couple wishes to work in the same geographic region. In such a situation, where couples have preferences over the pairs of jobs, it is known that stable outcomes do not always exist. The existing literature, Roth (1984) and Klaus and Klijn (2005), demonstrate this practically important fact by means of examples that have no stable outcomes.

Those examples are not quite easy to understand, however, because the non-existence is shown by checking a large number of outcomes one by one. Basically, we must check that either (a) or (b) is violated in each outcome.

There have also been some attempts to find sufficient conditions for the existence of a stable outcome in the presence of couples. Klaus and Klijn (2003) show that if all workers' preferences satisfy *responsiveness*, a stable outcome exists. In a subsequent work they claim that a weaker condition, *weak responsiveness*, is sufficient for the existence of a stable outcome (Klaus and Klijn (2005)).

In this paper, we offer a fairly simple example of couples' preferences under which no stable outcome exists (Example 1). The example itself is much simpler than the existing ones, and, more importantly, one can easily see the non-existence in a few lines of argument due to its symmetric structure. This also helps one to understand clearly which aspects of the example is crucial for the non-existence. Finally, compared to the original example by Roth (1984) where couples' preferences are somewhat arbitrary, the example in the present paper admits a simple intuitive interpretation.

The same example reveals that weak responsiveness defined by Klaus and Klijn is not a sufficient condition for the existence of a stable outcome, because it satisfies weak responsiveness (Claim 1). We point out the problem in Klaus and Klijn's proof (2005, Theorem 3.3) and introduce the notion of *reasonable responsiveness* as a weaker sufficient condition than the responsiveness (Theorem 1). Also we prove that reasonable responsiveness is the maximal condition for the existence (Theorem 2).

This paper is organized as follows. In Section 2, we introduce a labor market model with hospitals and couples of medical students, and some theorems about instability. In Section 3, we give an example that has no stable outcomes and examine the proof of Theorem 3.3 in Klaus and Klijn (2005).

[1] See Gale and Shapley (1962).

In Section 4, we introduce the notion of reasonable responsiveness and prove that this is a sufficient condition weaker than responsiveness, and we also show that it is the maximal condition for stability. We conclude with Section 5.

2 Matching Model with Couples

2.1 The model

I adopt the same notation as in Klaus and Klijn (2005) (it originates with Roth and Sotomayor (1990)), which formalizes matching between hospitals and couples of medical students.

Let $H = \{h_1, h_2, \ldots, h_N\}$, $S = \{s_1, s_2, \ldots, s_{2M}\}$, and $C = \{c_1, c_2, \ldots, c_M\} = \{(s_1, s_2), (s_3, s_4), \ldots, (s_{2M-1}, s_{2M})\}$ be the set of hospitals, students, and couples. Each $h \in H$ has one position to be filled.Each $h \in H$, $s \in S$, and $c \in C$ has a strict,[2] transitive and complete preference relation $\succeq_h$, $\succeq_s$, and $\succeq_c$ over $S \cup \{\emptyset\}$, $H \cup \{u\}$, and all feasible elements in $[H \cup \{u\}]^2$ respectively. Here u and $\emptyset$ indicate being unmatched. Preferences can also be described as $P(h) = s_1, s_2, s_3, s_4, \emptyset$, which means that h prefers s_i to s_{i+1} for $i = 1, 2, 3$ and also prefers s_4 to being unemployed. Let $P^H = (P(h))_{h \in H}$ be the profile of hospitals' preferences. $P(s), P^S, P(c), P^C$ are defined in a similar way.

(P^H, P^C) represents a matching market with couples. A *matching* μ for (P^H, P^C) is an assignment of students and hospitals. A match for couple $c = (s_k, s_l)$ is $\mu(c) = (\mu(s_k), \mu(s_l))$, where $\mu(s_i)$ is either a hospital or u, for $i = k, l$. Similarly, a hospital's match $\mu(h)$ is either a student or $\emptyset$.

A matching $\mu(\cdot)$ is *individually rational* if
(i1) for all $c = (s_k, s_l) \in C$, $(\mu(s_k), \mu(s_l)) \succeq_c (\mu(s_k), u)$, $(\mu(s_k), \mu(s_l)) \succeq_c (u, \mu(s_l))$, and $(\mu(s_k), \mu(s_l)) \succeq_c (u, u)$,
(i2) for all $h \in H$, $\mu(h) \succeq_h \emptyset$.

For a given matching μ, $(c = (s_k, s_l), (h_p, h_q))$ is a *blocking coalition* if
(b1) $(h_p, h_q) \succ_c (\mu(s_k), \mu(s_l))$,
(b2) $[h_p \in H \Rightarrow s_k \succeq_{h_p} \mu(h_p)]$ and $[h_q \in H \Rightarrow s_l \succeq_{h_q} \mu(h_q)]$.

A matching $\mu(\cdot)$ is *stable* if it is individually rational and if there are no blocking coalitions.

2.2 Existing Results

In this subsection we summarize important existing results.

[2] For example, $s \succeq_h s'$ and $s' \succeq_h s$ are satisfied if and only if $s = s'$.

Theorem 10 in Roth (1984). *In a market in which some agents are couples, a stable outcome do not always exist.*

Roth (1984) provides the first example of preferences of couples that has no stable outcomes. To see the non-existence in this example, one need to check all 24 possible outcomes one by one. Moreover, the couples' preferences are fairly complex and seem to be somewhat arbitrary (see Appendix A).[3]

Following Roth (1984), Klaus and Klijn further investigate the matching problem with couples. Their agenda include finding sufficient conditions for the existence of stable outcomes in the presence of couples. In an early version of their study (Klaus and Klijn (2003)), they introduced the following notion:

Definition. Couple $c = (s_k, s_l)$ has *responsive preferences (responsiveness)* if there exist preferences $\succ_{s_k}$ and $\succ_{s_l}$ such that for all $h_p, h_q, h_r \in H \cup \{u\}$,

$$h_p \succ_{s_k} h_r \Rightarrow (h_p, h_q) \succ_c (h_r, h_q), \quad h_p \succ_{s_l} h_r \Rightarrow (h_q, h_p) \succ_c (h_q, h_r).$$

Then, they proved

Theorem 3.1 in Klaus and Klijn (2003). *Let (P^H, P^C) be a couples market where couples have responsive preferences. Then, any matching that is stable for the associated singles market $(P^H, P^S(P^C))$ is also stable for (P^H, P^C). In particular, there exists a stable matching for (P^H, P^C).*[4]

A subsequent version, Klaus and Klijn (2005), introduced *weak responsiveness* as a weaker notion of responsiveness.

Definition. Couple $c = (s_k, s_l)$ has *weakly responsive preferences (weak responsiveness)* if there exist preferences $\succ_{s_k}$ and $\succ_{s_l}$ such that

1. for all $h \in H$, $(h, u) \succ_c (u, u) \Leftrightarrow h \succ_{s_k} u$ and $(u, h) \succ_c (u, u) \Leftrightarrow h \succ_{s_l} u$.

2. for all $h_p, h_q, h_r \in H \cup \{u\}$,

$$\begin{aligned} h_p \succeq_{s_k} u, \quad h_q \succeq_{s_l} u, \quad \text{and} \quad h_p \succ_{s_k} h_r \Rightarrow (h_p, h_q) \succ_c (h_r, h_q), \\ h_p \succeq_{s_k} u, \quad h_q \succeq_{s_l} u, \quad \text{and} \quad h_q \succ_{s_l} h_r \Rightarrow (h_p, h_q) \succ_c (h_p, h_r). \end{aligned}$$

[3]Examples 3.8 and 3.9 in Klaus and Klijn (2005) also show the instability and the preferences of couples. However, in these examples one need to check all outcomes one by one to show the instability.

[4]We obtain $(P^H, P^S(P^C))$ by replacing couples and their preferences in P^C by individual students and their individual preferences $P^S(P^C)$ (individual preferences must satisfy the required condition of the responsiveness).

Then they try to extend Theorem 3.1 in Klaus and Klijn (2003) and present the following statement:

Theorem 3.3 in Klaus and Klijn (2005). *Let* (P^H, P^C) *be a couples market where couples have weakly responsive preferences. Then, any matching that is stable for an associated singles market* $(P^H, P^S(P^C))$ *is also stable for* (P^H, P^C)*. In particular, there exists a stable matching for* (P^H, P^C)*.*

3 A Simple Example of Instability with Couples

In this section we introduce a simple example of couples' preferences under which no stable outcome exists. The example is much simpler than existing ones, and thanks to its symmetric structure one can easily see why no stable outcome exists. This example also shows that Theorem 3.3 in Klaus and Klijn (2005) is not correct.

Example 1. The example is shown in Table 1. Unlike the original example of instability by Roth (Appendix A), where couples' preferences are somewhat arbitrary, this example admits a plausible interpretation. Assume that hospitals h_1 and h_2 specialize in surgery, while h_3 and h_4 are notable for internal medicine. After their residency is over, couple (s_1, s_2) wants to practice in a specialized hospital (either in surgery or in internal medicine). In contrast, (s_3, s_4) plans to practice in a general hospital so that s_3 and s_4 need division of labor with different specialties. Thus the former couple wants to find jobs in hospitals with the same specialty, and the latter couple seeks hospitals with different specialties.

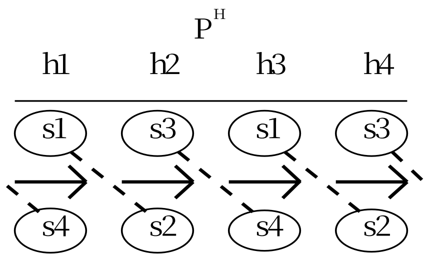

Figure 1: Blocking Coalitions

Table 1: Example 1.

P^H				P^C	
h_1	h_2	h_3	h_4	(s_1, s_2)	(s_3, s_4)
s_1	s_3	s_1	s_3	h_1, h_2	h_2, h_3
s_4	s_2	s_4	s_2	h_3, h_4	h_4, h_1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	u, u	u, u

Table 2: Outcomes and Blocking Coalition

outcome	h_1	h_2	h_3	h_4	blocking coalition
1	s_1	s_2	$\emptyset$	$\emptyset$	$(s_3, s_4), (h_2, h_3)$
2	$\emptyset$	s_3	s_4	$\emptyset$	$(s_1, s_2), (h_3, h_4)$
3	$\emptyset$	$\emptyset$	s_1	s_2	$(s_3, s_4), (h_4, h_1)$
4	s_4	$\emptyset$	$\emptyset$	s_3	$(s_1, s_2), (h_1, h_2)$

Proposition 1. *Example 1 has no stable outcome.*

Proof. Let $N = 0, 1, 2, 3, 4$ be the number of students who are employed. We show that for each $N = 0, \cdots, 4$, every possible outcome is either not individually rational or has some blocking coalition.

$N = 0$: Some couples and hospitals form a blocking coalition (ex. $c_1, (h_1, h_2)$).

$N = 1, 3$: There must be a couple in which only one member is employed. This couple would prefer being both unemployed to this situation. Thus this outcome is not individually rational.

$N = 2$: This is the essential part of this example. There are only 4 outcomes that are individually rational, where only one couple is fully employed, but each of the 4 outcomes has a blocking coalition. For example $(h_1, h_2), (s_1, s_2)$ has a blocking coalition $(h_2, h_3), (s_3, s_4)$. Symmetric explanations apply to other cases. Table 2 summarizes the blocking coalition for the case of $N = 2$. Figure 1 schematically shows the structure of these blocking coalitions. In the figure, each dotted line represents an individually rational matching, and the arrows indicate how the blocking coalitions work.

$N = 4$: (s_1, s_2) must be matched to either (h_1, h_2) or (h_3, h_4) in order to be individually rational. Then (s_3, s_4) should be matched to the rest, (h_3, h_4) or (h_1, h_2), but neither of them are individually rational for (s_3, s_4). $\square$

The symmetric structure of the preferences in Example 1 makes it easier to understand the instability than Roth's example (Table 3 in Appendix

A) or Klaus and Klijn's examples (Examples 3.8 or 3.9 in Klaus and Klijn (2005))[5] whose instability must be checked one by one.

One may argue that Example 1 is not quite realistic because of our assumption $(u, u) \prec_c (u, h)$. That is to say, the couples in our example prefer being both unemployed to only one member's being employed. It is not, however, the essential part of our example. Appendix B shows that we can readily introduce unemployment aversions in our example.

Next we contrast this example with the sufficient condition reported in Klaus and Klijn (2005).

Claim 1. *Example 1 satisfies weak responsiveness.*

Proof. We assume $P(s_i) = u, \cdots$ $(i = 1, 2, 3, 4)$. We check weak responsiveness (i) and (ii).

(i) For each $c \in C$, there exist no $h \in H$ such that $(h, u) \succ_c (u, u)$ or $(u, h) \succ_c (u, u)$ exist. For each $s \in S$, there exist no $h \in H$ such that $h \succ_s u$ exists. Thus, Example 1 satisfies the condition of (i) for all $h \in H$.

(ii) For each $s \in S$, u is the only one h which satisfies $h \succeq_s u$. Thus $h_p = u$, $h_q = u$. Any $h_r \in H$ satisfies $h_p(= u) \succ_{s_k} h_r$ and $h_q(= u) \succ_{s_l} h_r$. For each $c \in C$, Any $h_r \in H$ satisfies $(h_p(= u), h_q(= u)) \succ_{s_k} (h_r, h_q(= u))$ and $(h_p(= u), h_q(= u)) \succ_{s_l} (h_p(= u), h_r)$. Thus, Example 1 satisfies the condition of (ii) for all $h_r \in H$. □

Hence, Example 1 is a counterexample for Theorem 3.3 in Klaus and Klijn (2005), since this example satisfies weak responsiveness (Claim 1) and has no stable outcomes (Proposition 1).[6] Now we point out where the the proof of Theorem 3.3 in Klaus and Klijn (2005) had a problem. First, we show the outline of their proof.

The Outline of the Proof of Theorem 3.3 in Klaus and Klijn (2005)

1. Let μ be a stable matching for $(P^H, P^S(P^C))$.
2. μ is individually rational for (P^H, P^C) as well.

[5] Klaus and Klijn (2005) also prove that their examples have stable outcomes under the notion of weak stability in full employment. As for "weak stability", see Klaus and Klijn(2005), and Klijn and Massó (2003). One can easily see that our example also has a stable outcome under weak stability.

[6] One can also modify Roth's example to obtain a counter example to Klaus and Klijn's claim. Roth's example only specifies a part of each couple's preferences, which suffices to show instability. The unspecified parts correspond to the relative rankings of $(u, u), (h, u)$, and $(u, h), h \in H$, and it is not difficult to specify those parts of preferences to satisfy the weak responsiveness.

3. Suppose μ is not stable for (P^H, P^C). There must exist blocking coalition $(h_p, h_q), (s_k, s_l)$ which satisfies (b1) and (b2).

3.1 Suppose $h_p \prec_{s_k} u$, $h_q \prec_{s_l} u$. It invites contradiction.

3.2 Suppose $h_p \succeq_{s_k} u$, $h_q \prec_{s_l} u$. (h_p, u) (satisfies 3.4) is blocking coalition.

3.3 Suppose $h_p \prec_{s_k} u$, $h_q \succeq_{s_l} u$. (h_p, u) (satisfies 3.4) is blocking coalition.

3.4 Suppose $h_p \succeq_{s_k} u$, $h_q \succeq_{s_l} u$. It invites contradiction.

4. A stable matching for $(P^H, P^S(P^C))$ always exist (Gale and Shapley (1962)).

We point out that there is a flaw in Step 3.1.

3.1 (original text). Assume $h_p \prec_{s_k} u$ and $h_q \prec_{s_l} u$. Then by weak responsiveness (ii), $(u, u) \succ_c (u, h_q) \succ_c (h_p, h_q)$. Using (b1) it follows that $(u, u) \succ_c (\mu(s_k), \mu(s_l))$, contradicting individual rationality of μ in (P^H, P^C).

Here $(u, h_q) \succ_c (h_p, h_q)$ does not follow from weak responsiveness (ii). That is because, $h_q \succeq_{s_l} u$, which is necessary to use weak responsiveness (ii), is not satisfied by the assumption.

Since the correct sufficient condition, responsiveness (Theorem 3.1 in Klaus and Klijn (2003)), is a rather strong requirement, we introduce a weaker sufficient condition in the following section.

4 Reasonably Responsive Preferences

In this section, we introduce *reasonably responsive preferences*, which excludes Example 1, as a sufficient condition for stability.[7] Also we prove that reasonable responsiveness is a weaker notion of responsiveness. Lastly, we conclude that reasonable responsiveness defines the maximal domain of preferences for the existence of a stable outcome.

Definition. Couple $c = (s_k, s_l)$ has *reasonably responsive preferences (reasonable responsiveness)* if there exist preferences $\succ_{s_k}$ and $\succ_{s_l}$ such that

[7] The following considerations motivate our definition. The individual preferences constructed to show the weak responsiveness in Example 1 might not be so natural. Despite that $P(c) = (h_1, h_2), (h_3, h_4), (u, u)$, individual preferences satisfy $P(s) = u, \dots$. It might not be so reasonable that neither of h_1 nor h_2 is wanted by the individual. Thus we try to exclude such a case.

(i) For all $h_p, h_q \in H \cup \{u\}$, $(h_p, h_q) \succ_c (u, u) \Rightarrow h_p \succ_{s_k} u$ or $h_q \succ_{s_l} u$.

(ii) For all $h_p, h_q, h_r \in H \cup \{u\}$,

$$h_p \succeq_{s_k} u, \quad h_q \succeq_{s_l} u, \quad h_p \succ_{s_k} h_r \quad \Rightarrow \quad (h_p, h_q) \succ_c (h_r, h_q),$$
$$h_p \succeq_{s_k} u, \quad h_q \succeq_{s_l} u, \quad h_q \succ_{s_l} h_r \quad \Rightarrow \quad (h_p, h_q) \succ_c (h_p, h_r).$$

Then we prove that reasonable responsiveness is a sufficient condition for the existence of a stable outcome.

Theorem 1. *Let (P^H, P^C) be a couples market where couples have reasonable responsive preferences. Then, any matching that is stable for an associated singles market $(P^H, P^S(P^C))$ is also stable for (P^H, P^C). In particular, there exists a stable matching for (P^H, P^C).*

Proof. We can apply the same proof as in Theorem 3.3 of Klaus and Klijn (2004) except for Step 3.1 since reasonable responsiveness is a stronger notion of weak responsiveness.We modify the Step 3.1 as follows:

Assume that $h_p \prec_{s_k} u$ and $h_q \prec_{s_l} u$. By weak responsiveness (i), $(u, u) \succ_c (h_p, h_q)$. By (b1), $(h_p, h_q) \succ_c (\mu(s_k), \mu(s_l))$, so $(u, u) \succ_c (\mu(s_k), \mu(s_l))$. It contradicts the individual rationality of μ in (P^H, P^C). □

Then we prove that reasonable responsiveness is a weaker notion of responsiveness (Proposition 2 and Example 2).

Proposition 2. *Responsiveness implies reasonable responsiveness.*

Proof. Condition (ii) of reasonable responsiveness follows from responsiveness. Suppose that condition (i) is violated under responsiveness. Then there exist h_p, h_q such that $(h_p, h_q) \succ_c (u, u)$, $h_p \preceq_{s_k} u$, $h_q \preceq_{s_l} u$. By responsiveness, however, we have that $u \succeq_{s_k} h_p$, $u \succeq_{s_l} h_q \Rightarrow (u, u) \succeq_c (h_p, h_q)$. This contradicts $(h_p, h_q) \succ_c (u, u)$, hence condition (i) is satisfied under responsiveness. □

Example 2 (Reasonable responsiveness does not imply responsiveness). For $c = (s_1, s_2)$, $P(c) = (h_1, u), (h_2, u), (h_2, h_3), (h_1, h_3), (u, u), (u, h_3)$,we assume $P(s_1) = h_1, h_2, u, \ldots$, $P(s_2) = u, \ldots$. This example satisfies reasonable responsiveness. On the other hand, since $(h_1, u) \succ_c (h_2, u)$, $h_1 \succ_{s_1} h_2$ is needed in order to satisfy responsiveness. Also, since $(h_2, h_3) \succ_c (h_1, h_3)$, $h_2 \succ_{s_1} h_1$ is needed in order to satisfy responsiveness. This is a contradiction and hence it cannot satisfy responsiveness.

It would be natural to ask if it is also necessary. To obtain a tight necessary condition, however, we would generally need to consider the *relationship* between one couple's preferences and another's, which could be quite complex.

Instead, we focus on the *common* condition on the preferences of couples in the following way. Here we will introduce the notion of *maximality*:

Definition. The condition A is a *maximal condition* for the existence of a stable outcome if,

(i) "all couples' preferences satisfy A" is a sufficient condition, and,

(ii) there are no weaker sufficient condition B such that "all couples' preferences satisfy B" is a sufficient condition.

Klaus and Klijn (2005) use the notion of *maximal domain* as follows: A sufficient condition A is the maximal domain if, in a couples market with at least one couple whose preferences do not satisfy A, we can construct other couples' preferences satisfying A such that no stable matching exists. It is easy to show that the notions of maximal condition and maximal domain are equivalent. In the proofs, we use the notion of maximal domain, which is easier to check.

The following condition, due to Klaus and Klijn (2005), is a fairly natural requirement, and is implied by reasonable responsiveness.[8]

Definition. c is *restricted strictly unemployment averse* if for all $h_p, h_q \in H$ such that $(h_p, u) \succ_c (u, u)$ and $(u, h_q) \succ_c (u, u)$, both $(h_p, h_q) \succ_c (h_p, u)$ and $(h_p, h_q) \succ_c (u, h_q)$ are satisfied.

Under this condition, we show that reasonable responsiveness is the maximal condition on the preferences of couples for the existence of a stable outcome.[9]

Theorem 2. *Reasonable responsiveness is the maximal condition for the existence of a stable outcome under the restricted strictly unemployment aversion condition.*

See the Appendix C for the proof of Theorem 2. Here we give a rough sketch of the proof.

[8]The proof goes as follows: For all h_p, h_q such that $(h_p, u) \succ_c (u, u)$ and $(u, h_q) \succ_c (u, u)$, both $h_p \succ_{s_k} u$ and $h_q \succ_{s_l} u$ are satisfied by condition (i) of reasonable responsiveness. Thus by condition (ii) of reasonable responsiveness, we have $(h_p, h_q) \succ_c (h_p, u)$ and $(h_p, h_q) \succ_c (u, h_q)$.

[9]Klaus and Klijn (2005) also try to show weak responsiveness is a maximal domain under the restricted strictly unemployment aversion condition.

Step 1. We show that if there exists one couple whose most preferred pair of hospitals is either (h, u) or (u, h), and whose preference violates reasonable responsiveness but satisfies restricted strictly unemployment aversion, then we can construct other couples' preferences under the restriction of reasonable responsive preferences and hospitals' preferences in such a way that there exist no stable outcomes (Lemmas 1–2 and Corollary 1).

Step 2. We show the parallel statement as in Step 1 for the case in which each couple's most preferred pair is (h, h') $(h, h' \in H)$ (Lemmas 4–6).

5 Conclusion

This paper introduced an example (Example 1) which shows the possibility of non-existence of stable outcomes in a matching model with couples. Due to its simplicity and symmetry, the example enables us to understand the causes of the instability. By using this example, we showed that weak responsiveness is not a sufficient condition for the existence of a stable outcome. We proposed a new sufficient condition for preferences of couples, reasonable responsiveness, as a weaker notion of responsiveness that assures the existence of a stable outcome (Theorem 1). Moreover we proved that reasonable responsiveness is a maximal condition under the restriction of the restricted strictly unemployment aversion condition (Theorem 2).

Appendices

A. Roth's Example. Roth's example (1984, Theorem 10) is shown in Table 3. Each full employment outcome has a blocking coalition as shown in Table 4. The bold face indicates the one who causes this blocking. Here, one hospital and one member of a couple block each outcome. Other outcomes are blocked by some couples which include unemployed students.

B. Unemployment Aversion.

Definition. c is *strongly unemployment averse* if for all $h_p, h_q, h_r \neq u$, $(h_p, h_q) \succ_c (h_r, u) \succ_c (u, u)$ and $(h_p, h_q) \succ_c (u, h_r) \succ_c (u, u)$. c is *strictly unemployment averse* if for all $h_p, h_q \neq u$, $(h_p, h_q) \succ_c (h_p, u) \succ_c (u, u)$ and $(h_p, h_q) \succ_c (u, h_q) \succ_c (u, u)$.

Example 1 does not satisfy strong unemployment aversion, but this point is not essential for the non-existence of stable outcomes.

Proposition 3. *In Example 1, only top two hospitals in preferences of couples cause instability. In other words, even if we put additional hospitals in the subsequence of the preferences, Example 1 still remains unstable.*

Proof. Let $M = \{0, 1, 2\}$ be the number of students who are matched to hospitals as the first position of hospital's preferences. We show that under each M, every possible outcome has some blocking coalition.

$M = 2, 1$: There exist h_i and h_{i+1} such that $\mu(h_i)$ is the top of i's preference but $\mu(h_{i+1})$ is not that of $i + 1$'s.[10] For example, we consider the case in which $\mu(h_1) = s_1$ and $\mu(h_2) \neq s_3$. Then $c_2, (h_2, h_3)$ form a blocking coalition.

$M = 0$**:** $c_2, (h_2, h_3)$ form a blocking coalition. □

Proposition 3 assures that instability in Example 1 will be kept even if we put additional hospitals in the subsequence of preferences in the example. Thus we can arbitrarily construct a modified version of Example 1 so that it satisfies strong unemployment aversion conditions and has no stable outcome.

However, we "must" lose weak responsiveness.

Proposition 4. *Weakly responsive preferences in which all hospitals are acceptable are responsive.*

Proof. By condition (i) of weak responsiveness, we have $h_p \succ_{s_k} u$ and $h_q \succ_{s_l} u$ for all $h_p, h_q \in H$. By condition (ii) of weak responsiveness, for all $h_p, h_q \in H \cup \{u\}$, we have

$$h_p \succ_{s_k} h_r \Rightarrow (h_p, h_q) \succ_c (h_r, h_q), \quad \text{and} \quad h_q \succ_{s_l} h_r \Rightarrow (h_p, h_q) \succ_c (h_p, h_r).$$

□

C. Proof of Theorem 2.

Step 1. *If there exists one couple whose most preferred pair of hospitals is either (h, u) or (u, h), and whose preference violates reasonable responsiveness but satisfies restricted strictly unemployment aversion, then we can construct other couples' reasonably responsive preferences and hospitals' preferences in such a way that there exist no stable outcomes.*

Lemma 1. *If there exists h_i such that $(h_i, u) \succ_c (h_i, h_j)$ for all $h_j \neq u$ and that $(h_i, u) \succ_c (u, u)$, and if c satisfies restricted strictly unemployment aversion, then $(u, u) \succ_c (u, h_j)$ for all $h_j \neq u$.*

Proof. If $(u, h_j) \succ_c (u, u)$, restricted strictly unemployment aversion implies $(h_i, h_j) \succ_c (h_i, u)$. □

[10]We define h_5 as being h_1.

Lemma 2. *Suppose* $(u,u) \succ_c (u,h_j)$ *for all* $h_j \neq u$. *If* $(h_i,u) \succeq_c (h_i,h_j)$ *for all* h_i, h_j *such that* $(h_i,h_j) \succ_c (u,u)$, *then the couple* $c = (s_1,s_2)$ *satisfies reasonable responsiveness.*

Proof. Suppose that c satisfies the assumption. Then for all h_i, h_j such that $(h_i,h_j) \succ_c (u,u)$, we can define $P(s_1)$ as $h_i \succ_{s_1} u$, $u \succeq_{s_1} h_k$ for all $h_k \neq h_i$, and $(h_s,u) \succ_c (h_t,u) \Rightarrow h_s \succ_{s_1} h_t$. We also define $P(s_2) = u$. It satisfies reasonable responsiveness: (i) For all h_i, h_j such that $(h_i,h_j) \succ_c (u,u)$, $h_i \succ_{s_1} u$. (ii) h_q such that $h_q \succeq_{s_2} u$, is only u. h_r such that $h_p \succ_{s_1} h_r$, satisfies $(h_p,u) \succ_c (h_r,u)$. Thus, $h_p \succeq_{s_1} u$, $h_q \succeq_{s_l} u$, and $h_p \succ_{s_1} h_r \Rightarrow (h_p,h_q) \succ_c (h_r,h_q)$. h_p such that $h_p \succeq_{s_1} u$, is included in h_i. h_q such that $h_q \succeq_{s_2} u$, is only u. h_r such that $h_q (= u) \succ_{s_2} h_r$, can be any h. This h must be included in h_j or satisfies $(u,u) \succ (h_p,h)$. Thus, $h_p \succeq_{s_1} u$, $h_q \succeq_{s_l} u$, and $h_q \succ_{s_2} h_r \Rightarrow (h_p,h_q) \succ_c (h_p,h_r)$. □

In what follows, we use the notion of "part of preference". We make part of preferences of $P(c)$ by deleting $(h_{l_1},\cdot),(h_{l_2},\cdot),\ldots$ and/or $(\cdot,h_{m_1}),(\cdot,h_{m_2}),\ldots$.

Lemma 3. *If there exists a couple whose preference (or some part of whose preference) can be described as* $P(c) = (h_i,h_j),(**),(h_{i'},h_{j'}),\ldots$ *(*$h_i \neq u, h_{i'}$, $h_j \neq u, h_{j'}$, $(**)$ *is empty or some of* $(h_i,\cdot)$*), then we can construct a set of preferences which has no stable outcomes with other couples' preferences which are constructed arbitrarily under reasonable responsiveness and with hospitals' preferences which are constructed arbitrarily.*

Proof. We can define $P(c_1) = (h_1,h_3),(**),(h_2,h_4),\ldots$ ($h_1 \neq u, h_2$ and $h_3 \neq u, h_4$) without loss of generality. We also define P^H as $P(h_1) = s_3, s_1, \emptyset$, $P(h_2) = s_1, \emptyset$ (if $h_2 \neq u$), $P(h_3) = s_2, s_3, \emptyset$, $P(h_4) = s_2, \emptyset$ (if $h_4 \neq u$), P^C as $P(c_2) = (h_3,u),(h_1,u),(u,u)$.

Suppose that μ is a stable outcome. Then $\mu(c_2)$ should be either (h_3,u), (h_1,u), or (u,u). If $\mu(c_2) = (h_3,u)$, then μ can be blocked by $(h_1,h_3), c_1$ for $\mu(c_1) \neq (h_1,h_3)$. If $\mu(c_2) = (h_1,u)$, then μ can be blocked by h_3, c_2 for $\mu(c_2) \neq (h_3,u)$ and $\mu(c_1) = (h_2,h_4)$. If $\mu(c_2) = (u,u)$, then μ can be blocked by h_1, c_2 for $\mu(c_2) \neq (h_1,u)$. Thus, no stable outcome exists. □

Then we have the following corollary.

Corollary 1. *If there exists a couple whose most preferred pair of hospitals is* (h_i,h_j) *(*$h_i, h_j \in H$*), and whose preference satisfies either* $(u,u) \succ (h_i,u)$ *or* $(u,u) \succ (u,h_j)$*, then we can construct other couples' preferences which are constructed arbitrarily under reasonable responsiveness and hospitals' preferences which are constructed arbitrarily, so that there exist no stable outcomes.*

We complete Step 1 by using Lemmas 1–2 and Corollary 1:

Proof of Step 1. Suppose that there exists couple c whose most preferred pair of hospitals is (h, u) and whose preference violates reasonable responsiveness but satisfies restricted strictly unemployment aversion. That most preferred pair of hospitals of c is (h, u), and that c satisfies restricted strictly unemployment aversion imply $(u, u) \succ_c (u, h_j)$ for all $h_j \neq u$ (Lemma 1). If $(h_i, u) \succeq_c (h_i, h_j)$ for all h_i, h_j such that $(h_i, h_j) \succ_c (u, u)$, then it contradicts that the couple violates reasonable responsiveness (Lemma 2). Thus, it must be that $(h_i, h_j) \succ_c (h_i, u)$ for some h_i, h_j such that $(h_i, h_j) \succ_c (u, u)$. Thus there must exist h_i, h_j such that $(h_i, h_j) \succ_c (u, u)$, $(h_i, h_j) \succ_c (h_i, u)$, and $(u, u) \succ_c (u, h_j)$. By setting $P(h_k) = \emptyset$ for $h_k \neq h_i, h_j$, $\emptyset \prec_{h_i} s_1$ and $\emptyset \prec_{h_j} s_2$, we can apply Corollary 1, and conclude that we can construct the set of preferences which has no stable outcomes. □

Step 2. *Suppose that "if there exists one couple whose most preferred pair of hospitals is (h, h') $(h, h' \in H)$, and whose preference violates reasonable responsiveness but satisfies restricted strictly unemployment aversion, then we can construct other couples' preferences under the restriction of reasonable responsive preferences and hospitals' preferences so that there exist no stable outcomes." is not true, then it contradicts the conclusion of Step* 1 *(Lemmas* 4–6*).*

In what follows, we use $g(h_i, j), n(h_i)$ as follows: when we pick up all of $(h_i, \cdot)$ from $P(c)$ keeping the order of them, we can describe it as

$$(h_i, g(h_i, 1)), (h_i, g(h_i, 2)), \ldots, (h_i, g(h_i, n(h_i))), (h_i, u), \ldots.$$

Lemma 4. *If couple $c = (s_1, s_2)$ whose most preferred pair can be described as $(h_1, g(h_1, 1))$ $(h_1, g(h_1, 1) \neq u)$ and whose most preferred pair without $(h_1, \cdot)$ can be described as $(h_2, g(h_2, 1))$, and other couples' preferences which are constructed arbitrarily under reasonable responsiveness and hospitals' preferences which are constructed arbitrarily, cannot form the set of preferences which has no stable outcomes, then $(h_1, u) \succ_c (u, u)$, $(h_2, u) \succ_c (u, u)$, $g(h_1, l) = g(h_2, l)$, and $(h_1, g(h_1, l)) \succ_c (h_2, g(h_2, l))$ hold for $l = 1, 2, \ldots, n(h_1)$.*

Proof. From Corollary 1 we have $(h_1, u) \succ_c (u, u)$ and $(h_2, u) \succ_c (u, u)$.

In the following we prove $g(h_1, l) = g(h_2, l)$ and $(h_1, g(h_1, l)) \succ_c (h_2, g(h_2, l))$ for $l = 1, 2, \ldots, n(h_1)$.

$\underline{l = 1}$: It is obvious that $(h_1, g(h_1, 1)) \succ_c (h_2, g(h_2, 1))$. If $g(h_1, 1) \neq g(h_2, 1)$, then we can construct instability (Lemma 3). Thus $g(h_1, 1) = g(h_2, 1)$.

$\underline{l = k+1}$: Suppose that $g(h_1, l) = g(h_2, l)$ and $(h_1, g(h_1, l)) \succ_c (h_2, g(h_2, l))$ stand for $l = k$. Suppose also $s_1 \succ_{h_1} \emptyset$, $s_1 \succ_{h_2} \emptyset$ (if $h_2 \neq u$), $\emptyset \succ_{h_i} s_1$ ($i \neq 1, 2$), and $\emptyset \succ_{g(h_1,j)} s_2$ ($j = 1, 2, \ldots, k$). Most preferred and feasible set by c in this case would be $(h_1, g(h_1, k+1))$, since otherwise we can construct an instable example by changing $P(g(h_1, 1))$ as $s_2 \succ_{g(h_1,1)} \emptyset$ (Lemma 3). **(A)**

Most preferred and feasible set by c in the same case except for $(h_1, \cdot)$ must be included in $\{(h_2, g(h_2, k+1)), (u, \cdot)\}$ since $g(h_2, j) = g(h_1, j)$ ($j = 1, 2, \ldots, k$). Suppose that it is not $(h_2, g(h_2, k+1))$, which means $h_2 \neq u$. Then there must exists (u, f) ($f \neq g(h_1, j)$ for $j = 1, 2, \ldots, k$) such that $(u, f) \succ_c (h_2, g(h_2, k+1))$. In that case we can construct an instable example by setting $s_1 \succ_{h_2} \emptyset$, $\emptyset \succ_{h_i} s_1$ ($i \neq 2$), $s_2 \succ_{g(h_1,1)} \emptyset$ and $\emptyset \succ_{g(h_1,j)} s_2$ ($j = 2, \ldots k$) (Lemma 3). Thus it is $(h_2, g(h_2, k+1))$. **(B)**

(A) and **(B)** implies $g(h_2, k+1) = g(h_1, k+1)$ since $g(h_1, k+1) \neq u$ implies that we can construct an instable example (Lemma 3). Also **(A)** implies $(h_1, g(h_1, k+1)) \succ_c (h_2, g(h_2, k+1))$. Thus for $l = 1, 2, \ldots, n(1)$, $g(h_1, l) = g(h_2, l), (h_1, g(h_1, l)) \succ_c (h_2, g(h_2, l))$. $\square$

Lemma 5. *Suppose that couple $c = (s_1, s_2)$ whose most preferred pair can be described as $(h_1, g(h_1, 1))$ ($h_1, g(h_1, 1) \neq u$), and whose most preferred pair without $(h_1, \cdot)$ can be described as $(h_2, g(h_2, 1))$, violates reasonable responsiveness but satisfies restricted strictly unemployment aversion. Suppose also that it can form a set of preferences which has a stable outcome with any type of other couples' preferences which are constructed arbitrarily under reasonable responsiveness and hospitals' preferences which are constructed arbitrarily. Then we have $g(h_2, n(h_1) + 1) = u$ and $(h_1, u) \succ_c (h_2, u)$.*

Proof. Suppose on the contrary that $g(h_2, n(h_1) + 1) \neq u$. We consider $P'(c)$ which can be constructed by removing $(\cdot, g(h_1, i))$ ($i = 1, 2, \ldots, n(h_i)$) from $P(c)$.

Most preferred and feasible set $((h_1, \cdot), (h_2, \cdot), (u, \cdot))$ by c in $P'(c)$ is (h_1, u). This is because, supposing $s_1 \succ_{h_1} \emptyset$, $s_1 \succ_{h_2} \emptyset$ (if $h_2 \neq u$), $\emptyset \succ_{h_i} s_1$ ($i \neq 1, 2$), $s_2 \succ_{g(h_1,1)}$, and $\emptyset \succ_{g(h_1,j)} s_2$ ($j = 2, \ldots n(h_1)$) invites instable example in $P(c)$ (Lemmas 3–4).

$P'(c)$ satisfies restricted strictly unemployment aversion since $P(c)$ satisfies the same property. $P'(c)$ has a stable outcome since $P(c)$ has one even in the case of $\emptyset \succ_{g(h_1,i)} s_2$ for all $i = 1, 2, \ldots, n(h_i)$.

If $P'(c)$ satisfies reasonable responsiveness, then $(h_1, u) \succ_c (u, u)$ implies $h_1 \succ_{s_1} u$. Therefore $(h_1, u) \succ_c (h_1, g(h_2, n(h_1) + 1))$ implies $u \succ_{s_2} g(h_2, n(h_1) + 1)$. Also, $(h_2, u) \succeq_c (u, u)$ implies $h_2 \succeq_{s_1} u$, which implies that $(h_2, u) \prec_c (h_2, g(h_2, n(h_1) + 1))$ implies $u \prec_{s_2} g(h_2, n(h_1) + 1)$. From the contradiction we see that $P'(c)$ violates reasonable responsiveness.

Therefore $P'(c)$ must be able to form an example which has no stable outcomes because of Step 1. This is a contradiction. Thus it must be that $g(h_2, n(h_1) + 1) = u$ and $(h_1, u) \succ_c (h_2, u)$, following the similar discussion as in the proof of Lemma 4. □

Lemma 6. *Suppose that $P(c)$ whose most preferred pair can be described as $(h_1, g(h_1, 1))$ ($h_1, g(h_1, 1) \neq u$) and whose most preferred pair without $(h_1, \cdot)$ can be described as $(h_2, g(h_2, 1))$ violates reasonable responsiveness but satisfies restricted strictly unemployment aversion. Suppose also that it can form a set of preferences which has a stable outcome with any other couples' preferences which are constructed arbitrarily under reasonable responsiveness and any hospitals' preferences which are constructed arbitrarily. Then $P_2(c)$ which can be constructed by removing $(h_i, \cdot)$ from $P(c)$ satisfies the same property as that of $P(c)$.*

Proof. Firstly we have that $(h_1, u) \succ_c (u, u)$ and $(h_2, u) \succeq_c (u, u)$, which follows from the same logic as in Lemma 4.

$P_2(c)$ satisfies restricted strictly unemployment aversion since $P(c)$ satisfies the property. $P_2(c)$ has a stable outcome since $P(c)$ has one even in the case of $\emptyset \succ_{h_1} s_1$.

Lemmas 4–5 imply $g(h_1, l) = g(h_2, l)$ and $(h_1, g(h_1, l)) \succ_c (h_2, g(h_2, l))$ for $l = 1, 2, \ldots, n(h_1), n(h_1) + 1$. **(C)**

In what follows we show that $P_2(c)$ violates reasonable responsiveness. Suppose on the contrary that $P_2(c)$ is a reasonably responsive preference. Then there exist $P_2(s_1)$ and $P_2(s_2)$. We use $P(s_1) = h_1, P_2(s_1)$,$P(s_2) = P_2(s_2)$. As $h_1 \succ_{s_1} u$ in $P(c)$, $P(c)$ satisfies condition (i) of reasonable responsiveness (see Page 8).

For all h_q such that $h_q \succeq_{s_2} u$, $(h_2, h_q) \succeq_c (h_2, u)$ in $P_2(c)$ because of the reasonable responsiveness of $P_2(c)$. Thus $(h_1, h_q) \succ_c (h_2, h_q)$ holds in $P(c)$ from **(C)**. For all h_r such that $h_2 \succ_{s_1} h_r$, $(h_2, h_q) \succ_c (h_r, h_q)$ because of the reasonable responsiveness of $P_2(c)$. Thus $(h_1, h_q) \succ_c (h_r, h_q)$ in $P(c)$. Thus $P(c)$ satisfies the former condition of (ii) of reasonable responsiveness.

For all h_q such that $h_q \succ_{s_2} u$ and h_r such that $h_q \succ_{s_2} h_r$, we have $(h_2, h_q) \succ_c (h_2, h_r)$ in $P_2(c)$ because of the reasonable responsiveness of $P_2(c)$. Thus $(h_1, h_q) \succ_c (h_1, h_r)$ holds in $P(c)$ from **(C)**. Then $P(c)$ satisfies the latter condition of (ii) of reasonable responsiveness.

Thus $P(c)$ satisfies reasonable responsiveness, which is a contradiction. □

We complete Step 2 by using Lemmas 4–6.

Proof of Step 2. Suppose that a couple who prefers $(h_i, g(h_i, 1))$ $(h_i, g(h_i, 1) \neq u)$ to another, and another couple whose preferences are constructed arbitrarily under reasonable responsiveness and hospitals whose preferences are constructed arbitrarily, cannot form a set of preferences which has no stable outcomes. By Lemma 6 we can remove $(h_i, \cdot)$. By continuing the same way, we have that "one couple whose most preferred pair of hospital is (u, h) and whose preferences violates reasonable responsiveness but satisfies restricted strictly unemployment aversion, and another couple whose preferences satisfy reasonable responsiveness and hospitals always form the set of preferences which has a stable outcome." This contradicts Step 1. Thus we can form the set of preferences which has no stable outcomes. □

Lastly we complete the proof of Theorem 2.

Proof of Theorem 2. Suppose that reasonable responsiveness is not the maximal condition under restricted strictly unemployment aversion. Then there must be a sufficient condition B weaker than reasonable responsiveness. We should have a $P(c)$ which satisfies B but violates reasonable responsiveness since B is a weaker condition. Then $P(c)$ and other couples' preferences which are constructed arbitrarily under reasonable responsiveness and hospital's preferences which are constructed arbitrarily must have a stable outcome, since B is a sufficient condition for the existence of a stable outcome. However, Step 1 and Step 2 implies any of the $P(c)$ must produce a example which has no stable outcomes, since most preferred pair of the $P(c)$ cannot be (u, u) which implies that the preferences satisfy reasonable responsiveness. This contradiction implies reasonable responsiveness is the maximal condition under restricted strictly unemployment aversion. □

References

[1] Gale, D., and Shapley, L. S. (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly* **69**, 9–15.

[2] Klaus, B., and Klijn, F. (2005), "Stable Matchings and Preferences of Couples," *Journal of Economic Theory* **121**, 75–106.

[3] Klaus, B., and Klijn, F. (2003), "Stable Matchings and Preferences of Couples or Some Things Couples always Wanted to Know about Stable Matchings (but were afraid to ask)," Stanford Institute for Theoretical Economics (SITE) 2003 summer workshop, USA, August 10–18.

[4] Klijn, F., and Massó, J.(2003), "Weak Stability and a Bargaining Set for the Marriage Model," *Games and Economic Behavior* **42**, 91–100.

[5] Roth, A. E. (2002). "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," *Econometrica* **70**, 1341–1378.

[6] Roth, A. E., and Sotomayor, M. A. O. (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis* in Econometric Society Monograph Series. New York: Cambridge University Press.

Table 3: Roth's Example

P^H				P^C	
h_1	h_2	h_3	h_4	(s_1, s_2)	(s_3, s_4)
s_4	s_4	s_2	s_2	h_1, h_2	h_4, h_2
s_2	s_3	s_3	s_4	h_4, h_1	h_4, h_3
s_1	s_2	s_1	s_1	h_4, h_3	h_4, h_1
s_3	s_1	s_4	s_3	h_4, h_2	h_3, h_1
				h_1, h_4	h_3, h_2
				h_1, h_3	h_3, h_4
				h_3, h_4	h_2, h_4
				h_3, h_1	h_2, h_1
				h_3, h_2	h_2, h_3
				h_2, h_3	h_1, h_2
				h_2, h_4	h_1, h_4
				h_2, h_1	h_1, h_3

Table 4: All Full Employment Outcomes and Blocking Coalition

outcomes	h_1	h_2	h_3	h_4	blocking coalition
1	s_1	s_2	s_3	s_4	$(s_3, \mathbf{s_4}), h_2$
2	s_1	s_2	s_4	s_3	$(s_3, \mathbf{s_4}), h_2$
3	s_1	s_3	s_2	s_4	$(s_1, \mathbf{s_2}), h_4$
4	s_1	s_3	s_4	s_2	$(s_3, \mathbf{s_4}), h_1$
5	s_1	s_4	s_2	s_3	$(s_1, \mathbf{s_2}), h_4$
6	s_1	s_4	s_3	s_2	$(s_3, \mathbf{s_4}), h_1$
7	s_2	s_1	s_3	s_4	$(s_3, \mathbf{s_4}), h_1$
8	s_2	s_1	s_4	s_3	$(s_3, \mathbf{s_4}), h_1$
9	s_2	s_3	s_1	s_4	$(s_1, \mathbf{s_2}), h_4$
10	s_2	s_3	s_4	s_1	$(s_3, \mathbf{s_4}), h_1$
11	s_2	s_4	s_1	s_3	$(s_1, \mathbf{s_2}), h_4$
12	s_2	s_4	s_3	s_1	$(s_3, \mathbf{s_4}), h_1$
13	s_3	s_1	s_2	s_4	$(s_3, \mathbf{s_4}), h_2$
14	s_3	s_1	s_4	s_2	$(s_1, \mathbf{s_2}), h_3$
15	s_3	s_2	s_1	s_4	$(s_1, \mathbf{s_2}), h_4$
16	s_3	s_2	s_4	s_1	$(s_1, \mathbf{s_2}), h_3$
17	s_3	s_4	s_1	s_2	$(\mathbf{s_1}, s_2), h_1$
18	s_3	s_4	s_2	s_1	$(s_1, \mathbf{s_2}), h_1$
19	s_4	s_1	s_2	s_3	$(s_3, \mathbf{s_4}), h_2$
20	s_4	s_1	s_3	s_2	$(s_1, \mathbf{s_2}), h_3$
21	s_4	s_2	s_1	s_3	$(s_1, \mathbf{s_2}), h_4$
22	s_4	s_2	s_3	s_1	$(s_1, \mathbf{s_2}), h_3$
23	s_4	s_3	s_1	s_2	$(\mathbf{s_3}, s_4), h_3$
24	s_4	s_3	s_2	s_1	$(s_3, \mathbf{s_4}), h_4$