Pricing contingent claims with credit risk: Asymptotic expansion approach

YOSHIFUMI MUROI
Graduate School of Economics, University of Tokyo
Bunkyo-ku, Hongo 7-3-1, Tokyo 113-8654, JAPAN. e-mail:roomjing@ybb.ne.jp

Abstract. The pricing problem of credit derivatives has received much attention in the last decade. An important unresolved problem, however, is the pricing of credit derivatives under the general environment in which the interest rate process and the hazard rate process are stochastic. This article addresses the pricing problems of credit derivatives (defaultable bonds, default swaps, and default options) by the asymptotic expansion approach. This approach has only recently been introduced to mathematical finance, and it enables us to evaluate credit derivatives under the widely adapted class of models, including the affine structure models. A numerical study has also been presented.

Key words: defaultable bond, asymptotic expansion approach, spot interest rate, affine term structure, hazard rate process, credit defaultable swaps, options on defaultable bonds

1 Introduction

This paper provides pricing problems of credit derivatives using reduced modeling. See Bielecki and Rutkowski (2001) for precise review of the pricing problems of credit derivatives. There are mainly two kinds of approaches to the mathematical modeling of credit derivatives, viz. structural modeling and reduced modeling. Pricing problems of credit derivatives using structural modeling has been presented by Black and Cox (1976), for example.

In reduced modeling, the default time is regarded as an unpredictable, endogenous variable governed by the hazard rate process. The pricing problems of credit derivatives with reduced modeling were examined in Duffie and Singleton (1999), Jarrow and Turnbull (1995), Lando (1998), and Muroi (2002). Although there are many investigations on the theory of credit derivatives using the reduced modeling approach, most studies have assumed such features as the independence of the spot interest rate process and the hazard rate process, or the structure of the affine models. Duffie, Pan, and Singleton (2000) introduced the pricing methods under the affine models. However, their setting of the affine structure models seems sometimes restrictive. This paper demonstrates that it is possible to treat the pricing problems of credit derivatives under a widely adapted class of models by the asymptotic expansion approach. This method has recently been studied in the area of the stochastic analysis. See Ikeda and Watanabe (1989), for example. The asymptotic expansion approach has been applied to a wide range of topics such as statistics and mathematical finance, as in Kim and Kunitmo (1999), Kunitomo and Takahashi (2001, 2003a, 2003b), Takahashi (1999), and Yoshida (1992). There are two advantages of the asymptotic expansion approach. Firstly, it is possible to obtain the analytic approximation formula for the price of credit derivatives. This feature enables rapid computation of the price of contingent claims. Secondly, the figures derived by this method are very accurate. Although it is possible to justify this method using Malliavin calculus, which is known as the stochastic method of variations, only the formal asymptotic expansion methods will be examined in this paper. The validity of the asymptotic expansion approach has been shown by Ikeda and Watanabe (1989) and Kunitomo and Takahashi (2003a, 2003b).

Two important recovery rules are the RMV (recovery of market value) and the RT (recover of treasury) recovery rules. The RMV has been used by Duffie and Singleton (1999) and the RT by Jarrow and Turnbull (1995) and Muroi (2002), for example. In this paper, it will be shown that through use of the asymptotic expansion approach, it is possible to deal with the both recovery rules. For example, the default swaps whose underlying defaultable asset is following the RMV recovery rules and the default options with underlying defaultable asset with the RT recovery rules will be presented.

Section 2 focuses on the model of the defaultable bonds market, and provides the approximation formula for the price of defaultable bonds through the asymptotic expansion approach. The pricing problem of default swaps are considered in Section 3, and discussed the pricing problems of European put options on defaultable bonds in Section 4. Some numerical examples are given in Section 5, followed by some concluding remarks in Section 6.

2 A model of the defaultable bonds market

In this section, the defaultable bonds market will be formulated and the defaultable bond prices will be derived by the asymptotic expansion approach. This is considered in a frictionless economy with a finite trading horizon $[0, T]$.

The complete probability space $(\Omega, \mathcal{F}, P)$ which is formally large enough to support the positive valued continuous stochastic processes with a small parameter ϵ $(0 < \epsilon \leq 1)$, r_t^ϵ and h_t^ϵ, will be fixed. The probability measure, P, is considered as a real-world probability measure with the assumption that there is some risk neutral probability measure, Q, which is equivalent to the real-world probability measure P. It is possible to evaluate the price of credit derivatives based on the probability measure, Q. The positive valued stochastic process, r_t^ϵ, will be interpreted as the spot interest rate and the positive valued stochastic process, h_t^ϵ, will be interpreted as the hazard rate process. Next, the stopping time, τ, which will be regarded as the default time of the issuer of bonds will be introduced,

$$\tau = \inf\{t; \int_0^t h_s^\epsilon ds \geq E\} .$$

The random variable, E, is a unit exponential random variable which is independent of the 2-dimensional stochastic process $(r_t^\epsilon, h_t^\epsilon)$. The information sets at time t are given by

$$\begin{aligned}
\mathcal{G}_t &= \sigma\{(r_s^\epsilon, h_s^\epsilon)|0 \leq s \leq t\} \\
\mathcal{H}_t &= \sigma\{1_{\{\tau \leq s\}}|0 \leq s \leq t\} \\
\mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t .
\end{aligned}$$

Under these assumptions, we have following relationships:

$$\begin{aligned}
Q[\tau > t|\mathcal{G}_t] &= \exp(-\int_0^t h_s^\epsilon ds) \quad t \in [0, T] \\
Q[\tau > t] &= E^Q[\exp(-\int_0^t h_s^\epsilon ds)] \quad t \in [0, T] .
\end{aligned}$$

Lando (1998) proposed the division of securities into basic building blocks to evaluate defaultable contingent claims. The following lemma is due to Proposition 3.1 of Lando (1998, p104).

Lemma 2.1 *The random variable X satisfies $X \in \mathcal{G}_T$ and the $\mathcal{G}_t$ adapted stochastic process Z_t equals 0 after time T. By assuming that the expectations*

$$E^Q[\exp(-\int_t^T r_s^\epsilon ds)|X|] , \text{ and } E^Q[\int_t^T |Z_s \lambda_s| \exp(-\int_t^s r_u^\epsilon + h_u^\epsilon du)ds]$$

are finite, the following identities hold:

$$\begin{aligned}
E^Q[\exp(-\int_t^T r_s^\epsilon ds)X 1_{\{\tau > T\}}|\mathcal{F}_t] &= 1_{\{\tau > t\}} E^Q[\exp(-\int_t^T (r_s^\epsilon + h_s^\epsilon)ds)X|\mathcal{G}_t] \\
E^Q[\exp(-\int_t^\tau r_s^\epsilon ds)Z_\tau|\mathcal{F}_t] &= 1_{\{\tau > t\}} E^Q[\int_t^T Z_s h_s \exp(-\int_t^s (r_u^\epsilon + h_u^\epsilon)du)ds|\mathcal{G}_t]
\end{aligned}$$

There are two types of zero-coupon bonds in the market, default-free zero coupon bonds and defaultable zero coupon bonds. The former is a security paying \$1 at the maturity date, which is denoted by $p^\epsilon(t, u)$ with the maturity date, u ($u \leq T$), at time, t. The latter is a security whose pay-off depends on the recovery rules. Duffie and Singleton (1999) introduced several recovery rules for credit derivatives, some of which are listed below.

Assumption 2.1 *(recovery rules)*
i)(RMV(recovery of market value)) A defaultable zero coupon bond pays \$1 *at the maturity date if the issuer is solvent or pays* δ *times the defaultable bond price just before the time of bankruptcy.* $(0 \leq \delta < 1)$
ii)(RT(recover of treasury))A defaultable zero coupon bond pays \$1 *at the maturity date if the issuer is solvent or pays* \$$\delta$ *at the maturity date if bankruptcy has occurred before the maturity date.* $(0 \leq \delta < 1)$

The constant δ is termed the fractional recovery rate, and depends on the specific model which recovery rules should be used to evaluate defaultable bonds. In this paper, the defaulatble bond price will be evaluated with the recovery rules of both cases as examples. The RMV recovery rules will be used to evaluate credit default swaps in Section 3, and the RT recovery rules will be used to price options on defaultable bonds in Section 4. The RMV recovery rules of i) was used by Duffie and Singleton (1999) to evaluate a large class of credit derivatives, and the RT recovery rules of ii) was assumed by Jarrow and Turnbull (1995) and Muroi (2002) to evaluate, mainly, the price of options on defaultable bonds.

Example 2.1 i)Under the RMV recovery rules of Assumption 2.1-i), the zero coupon bond price at time, t, with the fractional recovery rate, δ, and the maturity date, T, is represented by $u^{\epsilon}_{\delta}(t,T)$. The pricing formula of this defaultable bond is given by

$$\begin{aligned} u^{\epsilon}_{\delta}(t,T) &= E^{Q}[\exp(-\int_t^T r^{\epsilon}_s ds)1_{\{\tau>T\}} + \exp(-\int_t^{\tau} r^{\epsilon}_s ds)(1-\delta)u^{\epsilon}_{\delta}(\tau-,T)1_{\{t<\tau\leq T\}}|\mathcal{F}_t] \\ &= 1_{\{\tau>t\}}E^{Q}[\exp(-\int_t^T r^{\epsilon}_s + (1-\delta)h^{\epsilon}_s ds)|\mathcal{G}_t] \ . \qquad (2.1) \end{aligned}$$

The stochastic process, $\tilde{u}^{\epsilon}_{\delta}$, is defined by $\tilde{u}^{\epsilon}_{\delta}(m,T) = E^{Q}[\exp(-\int_m^T (r^{\epsilon}_t + (1-\delta)h^{\epsilon}_t)dt)|\mathcal{G}_m]$. For the special case, the zero recovery defaultable zero coupon bond price $u^{\epsilon}_0(t,T)$ is denoted by $u^{\epsilon}(t,T)$. The stochastic process, $\tilde{u}^{\epsilon}_0(t,T)$, is also denoted by $\tilde{u}^{\epsilon}(t,T)$.
ii)Under the RMV recovery rules of Assumption 2.1-i), the coupon bond price at time t with the fractional recovery rate, δ, the maturity date, T, and the coupon rate, $b_i(>0)$, at the coupon payment date, u_i $(0 \leq u_1 \leq \cdots \leq u_n \leq T)$, is represented by $v^{\epsilon}_{\delta}(t,T)$. The pricing formula of this coupon bond is

$$v^{\epsilon}_{\delta}(t,T) = \sum_{u_i>t} b_i u^{\epsilon}_{\delta}(t,u_i) \ .$$

The new stochastic process $\tilde{v}^{\epsilon}_{\delta}(t,T)$ is also defined as $\tilde{v}^{\epsilon}_{\delta}(t,T) = \sum_{u_i>t} b_i \tilde{u}^{\epsilon}_{\delta}(t,u_i)$.
iii)Under the RT recovery rules of Assumption 2.1-ii), the zero coupon bond price at time, t, with the fractional recovery rate, δ, and the maturity date, T, is represented by $w^{\epsilon}_{\delta}(t,T)$. The pricing formula of this zero coupon bond is given by

$$w^{\epsilon}_{\delta}(t,T) = \delta p^{\epsilon}(t,T) + (1-\delta)u^{\epsilon}(t,T) \ ,$$

The new stochastic process is also defined as $\tilde{w}^{\epsilon}_{\delta}(t,T) = \delta p^{\epsilon}(t,T) + (1-\delta)\tilde{u}^{\epsilon}(t,T)$.

The pricing formula (2.1) in Example 2.1-i) can be found in Example 3.5 of Lando (1998, p107). The formula was proved by using the basic block building methods and Lemma 2.1. See also Duffie, Schroder, and Skiadas (1996).

The asymptotic expansion approach to the pricing problem of credit derivatives will be demonstrated here. The asymptotic expansion approach is a method which was introduced recently to mathematical finance by Kunitomo and Takahashi (2001). It is possible to justify the validity of the asymptotic expansion approach by using Malliavin calculus. The validity of this method has been shown by Ikeda and Watanabe (1989) and Kunitomo and Takahashi (2003a, 2003b).

Assumption 2.2 *On the equivalent martingale measure, Q, the non-negative valued spot interest rate process, r^{ϵ}_t, and the non-negative valued hazard rate process, h^{ϵ}_t, can be represented by the stochastic differential equation (SDE) as*

$$r^{\epsilon}_t = x_1 + \int_0^t \mu_1(\bar{x}_1 - r^{\epsilon}_s)ds + \sum_{j=1}^{2} \epsilon \int_0^t \sigma_{1j}(r^{\epsilon}_s, h^{\epsilon}_s)dW^j_s \qquad (2.2)$$

$$h^{\epsilon}_t = x_2 + \int_0^t \mu_2(\bar{x}_2 - h^{\epsilon}_s)ds + \sum_{j=1}^{2} \epsilon \int_0^t \sigma_{2j}(r^{\epsilon}_s, h^{\epsilon}_s)dW^j_s \ , \qquad (2.3)$$

where (W^1_t, W^2_t) is a two-dimensional normal Q-Brownian motion.

The following technical conditions are imposed to the stochastic processes r^{ϵ}_t and h^{ϵ}_t.

Assumption 2.3 *The volatility functions $\sigma_{ij}(r,h)$ (i,j=1,2) are $C^{\infty}_b(\mathbf{R}^2)$ class functions.*

Note that the affine term structure model does not satisfy assumption 2.3 in general. However, it is possible to justify the use of the affine term structure model by using smoothing arguments to the volatility functions. Moreover, a stochastic processes with the affine structure are not non-negative processes in general. Although the models in which the negative value of the stochastic interest rate and the hazard rate involved, such as Gaussian models, contradict the interpretation of the hazard rate process and the interest rate process, such a model is commonly used for the sake of the analytical tractability.

Example 2.2 (Affine term structure) The affine structure model is one of the most important for pricing interest rate derivatives and credit derivatives. Models become affine by providing some structure to the volatility functions:

$$\sigma_{i1}(x,y)\sigma_{j1}(x,y) + \sigma_{i2}(x,y)\sigma_{j2}(x,y) = H_{ij}(t) + K^1_{ij}(t)x + K^2_{ij}(t)y \ .$$

Note that we have relations, $H_{12} = H_{21},\ K^1_{12} = K^1_{21}$, and $K^2_{12} = K^2_{21}$. More information on the multi-factor affine structure models is available in Duffie (2001, p148).

Example 2.3 If there are some functions, $\sigma_1(\cdot)$, and, $\sigma_2(\cdot)$, such that

$$\sigma_{11}(x,y) = \sigma_1(x), \sigma_{12}(x,y) = 0, \sigma_{21}(x,y) = \rho\sigma_2(y),\ \textit{and}\ \sigma_{22}(x,y) = \sqrt{1-\rho^2}\sigma_2(y)\ ,$$

then the dynamics of the spot interest rate process and the hazard rate process are denoted as

$$\begin{aligned} r_t^\epsilon &= x_1 + \int_0^t \mu_1(\bar{x}^1 - r_s^\epsilon)ds + \epsilon \int_0^t \sigma_1(r_s^\epsilon)dW_s \\ h_t^\epsilon &= x_2 + \int_0^t \mu_2(\bar{x}^2 - h_s^\epsilon)ds + \epsilon \int_0^t \sigma_2(h_s^\epsilon)dZ_s\ , \end{aligned}$$

with $W = W^1$ and $Z = \rho W^1 + \sqrt{1-\rho^2}W^2$.

Example 2.3 is a simple and natural example, but has not been included in the affine structure models. It is possible, however, to derive the analytic approximation formula of Example 2.3 by the asymptotic expansion approach. The asymptotic expansion approach is applicable both to the RMV recovery rules of Assumption 2.1-i) and the RT recovery rules of Assumption 2.1-ii).

The zeroth order terms of the stochastic expansions for the stochastic processes, r_t^ϵ and h_t^ϵ, are denoted by X_t^1 and X_t^2. It is possible to derive these quantities by solving the integral equation below,

$$X_t^i = x_i + \int_0^t \mu_i(\bar{x}^i - X_s^i)ds \quad (i = 1, 2)\ . \tag{2.4}$$

The solutions of integral equation (2.4) are given by

$$X_t^i = \bar{x}_i - (\bar{x}_i - x_i)e^{-\mu_i t} \quad (i = 1, 2)\ .$$

The first and second order terms of the stochastic expansions for the stochastic processes, r_t^ϵ and h_t^ϵ, are represented by A_t^i and B_t^i $(i = 1, 2)$, and are given by

$$A_t^1 = \frac{\partial r_t^\epsilon}{\partial \epsilon}|_{\epsilon=0},\ A_t^2 = \frac{\partial h_t^\epsilon}{\partial \epsilon}|_{\epsilon=0}\ \textit{and}\ B_t^1 = \frac{1}{2}\frac{\partial^2 r_t^\epsilon}{\partial \epsilon^2}|_{\epsilon=0},\ B_t^2 = \frac{1}{2}\frac{\partial^2 h_t^\epsilon}{\partial \epsilon^2}|_{\epsilon=0}\ .$$

They satisfy the following equations,

$$\begin{aligned} A_t^i &= -\int_0^t \mu_i A_s^i ds + \sum_{j=1}^2 \int_0^t \sigma_{ij}(X_s^1, X_s^2)dW_s^j \\ B_t^i &= -\int_0^t \mu_i B_s^i ds + \sum_{j,l=1}^2 \int_0^t \partial_l \sigma_{ij}(X_s^1, X_s^2)A_s^l dW_s^j \quad (i = 1, 2)\ . \end{aligned}$$

The solutions of these equations are given by

$$A_t^i = \sum_{j=1}^2 \int_0^t Y_t^i (Y_s^i)^{-1} \sigma_{ij}(X_s^1, X_s^2)dW_s^j$$

$$
\begin{aligned}
B_t^i &= \sum_{j,l=1} \int_0^t Y_t^i (Y_s^i)^{-1} \partial_l \sigma_{ij}(X_s^1, X_s^2) A_s^l dW_s^j \\
&= \sum_{j,k,l=1} \int_0^t Y_t^i (Y_s^i)^{-1} \partial_l \sigma_{ij}(X_s^1, X_s^2) \int_0^s Y_s^l (Y_u^l)^{-1} \sigma_{lk}(X_u^1, X_u^2) dW_u^k dW_s^j \quad (i = 1, 2)
\end{aligned}
$$

where Y_t^i $(i = 1, 2)$ is the solution of the differential equation

$$
\frac{dY_t^i}{dt} = -\mu_i Y_t^i \text{ with } Y_0^i = 1 \quad (i = 1, 2)\ . \tag{2.5}
$$

The solution of the differential equation (2.5) is given by $Y_t^i = e^{-\mu_i t}$ $(i = 1, 2)$. The new random variables, $A_t^{ij} (i, j = 1, 2)$ and B_t^{ijk} $(i, j, k = 1, 2)$, are introduced by

$$
A_t^{ij} = \int_0^t \sigma_{ij}^A(t, s) dW_s^j \quad (i, j = 1, 2) \tag{2.6}
$$

$$
B_t^{ijk} = \sum_{l=1} \int_0^t \sigma_{ijl}^B(t, s) \int_0^s \sigma_{lk}^A(s, u) dW_u^k dW_s^j \quad (i, j, k = 1, 2) \tag{2.7}
$$

where

$$
\sigma_{ij}^A(t, s) = Y_t^i (Y_s^i)^{-1} \sigma_{ij}(X_s^1, X_s^2) \text{ and } \sigma_{ijk}^B(t, s) = Y_t^i (Y_s^i)^{-1} \partial_k \sigma_{ij}(X_s^1, X_s^2)\ .
$$

The formal Taylor expansions will lead to the stochastic expansion formulas for the stochastic processes, r_t^ϵ and h_t^ϵ, by

$$
\begin{aligned}
r_t^\epsilon &= X_t^1 + \epsilon A_t^1 + \epsilon^2 B_t^1 + o_Q(\epsilon^2) \\
h_t^\epsilon &= X_t^2 + \epsilon A_t^2 + \epsilon^2 B_t^2 + o_Q(\epsilon^2)\ .
\end{aligned}
$$

The default-free zero coupon bond price and the defaultable zero coupon bond price with the maturity date, T, at time, $m (0 \leq m \leq T)$, under the RMV recovery rules of Assumption 2.1-i) will now be evaluated . These prices are given by

$$
\begin{aligned}
p^\epsilon(m, T) &= E^Q[\exp(-\int_m^T r_t^\epsilon dt) | \mathcal{G}_m] \\
u_\delta^\epsilon(m, T) &= 1_{\{\tau > m\}} \tilde{u}_\delta^\epsilon(m, T) = 1_{\{\tau > m\}} E^Q[\exp(-\int_m^T r_t^\epsilon + (1 - \delta) h_t^\epsilon dt) | \mathcal{G}_m]\ .
\end{aligned}
$$

The following quantities should be estimated to derive the stochastic expansions of the stochastic processes, $p^\epsilon(m, T)$ and $u_\delta^\epsilon(m, T)$:

$$
\begin{aligned}
\int_m^T r_t^\epsilon dt &= \int_m^T X_t^1 dt + \epsilon \int_m^T A_t^1 dt + \epsilon^2 \int_m^T B_t^1 dt + o_Q(\epsilon^2) \\
\int_m^T h_t^\epsilon dt &= \int_m^T X_t^2 dt + \epsilon \int_m^T A_t^2 dt + \epsilon^2 \int_m^T B_t^2 dt + o_Q(\epsilon^2)\ .
\end{aligned}
$$

The first order terms of the stochastic expansions are calculated as

$$
\begin{aligned}
\int_m^T A_t^i dt &= \sum_{j=1} \int_m^T \int_0^t Y_t^i (Y_s^i)^{-1} \sigma_{ij}(X_s^1, X_s^2) dW_s^j dt \\
&= \sum_{j=1} (R_1^{ij}(m, T) + \tilde{R}_1^{ij}(m, T)) \quad (i = 1, 2)
\end{aligned} \tag{2.8}
$$

where

$$R_1^{ij}(m,T) = \int_0^m r_{ij}^A(T,m,s)dW_s^j, \quad \tilde{R}_1^{ij}(m,T) = \int_m^T r_{ij}^A(T,s,s)dW_s^j \quad (i,j=1,2)$$

with $r_{ij}^A(T,t,s) = (\int_t^T Y_u^i du)(Y_s^i)^{-1}\sigma_{ij}(X_s^1,X_s^2)$. Note that (2.8) has been derived by Fubini's theorem. The second order terms of the stochastic expansions are also calculated as

$$\begin{aligned}
\int_m^T B_t^i dt &= \sum_{j,k,l=1} \int_m^T \int_0^t Y_t^i (Y_s^i)^{-1} \partial_l \sigma_{ij}(X_s^1,X_s^2) \int_0^s \sigma_{lk}^A(X_u^1,X_u^2) dW_u^k dW_s^j dt \\
&= \sum_{j,k,l=1} (R_2^{ijk}(m,T) + \tilde{R}_2^{ijk}(m,T)) \quad (i=1,2)
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
R_2^{ijk}(m,T) &= \sum_{l=1} \int_0^m r_{ijl}^B(T,m,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j, \\
\tilde{R}_2^{ijk}(m,T) &= \sum_{l=1} \int_m^T r_{ijl}^B(T,s,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j \quad (i,j,k=1,2)
\end{aligned}$$

with $r_{ijk}^B(T,t,s) = (\int_t^T Y_u^i du)(Y_s^i)^{-1}\partial_k\sigma_{ij}(X_s^1,X_s^2)$. Fubibini's theorem was also used to derive (2.9).

It is possible to derive the price of default-free zero coupon bonds. It is given by

$$\begin{aligned}
p^\epsilon(m,T) &= E^Q[\exp(-\int_m^T r_t^\epsilon dt)|\mathcal{G}_m] \\
&= e^{-\int_m^T X_t^1 dt} e^{-\epsilon R_1^1(m,T) - \epsilon^2 R_2^1(m,T)} E^Q[e^{-\epsilon \tilde{R}_1^1(m,T) - \epsilon^2 \tilde{R}_2^1(m,T)}|\mathcal{G}_m] + o_Q(\epsilon^2) \\
&= e^{-\int_m^T X_t^1 dt}(1 - \epsilon R_1^1(m,T) - \epsilon^2 R_2^1(m,T) \\
&\quad + \frac{\epsilon^2}{2}(R_1^1(m,T))^2 + \frac{\epsilon^2}{2}\sum_{j=1}\int_m^T r_{1j}^A(T,t,t)^2 dt) + o_Q(\epsilon^2) .
\end{aligned} \tag{2.10}$$

where $R_1^i = \sum_{j=1}^2 R_1^{ij}$ $(i=1,2)$ and $R_2^i = \sum_{j,k=1}^2 R_2^{ijk}$ $(i=1,2)$. Note that the relations, $E^Q[\tilde{R}_1^1(m,T)|\mathcal{G}_m] = E^Q[\tilde{R}_2^1(m,T)|\mathcal{G}_m] = 0$, were used to derive (2.10). Similar calculations lead to the price of defaultable zero coupon bonds. The value of $\tilde{u}_\delta^\epsilon(m,T)$ is given by

$$\begin{aligned}
\tilde{u}_\delta^\epsilon(m,T) &= e^{-\int_m^T X_t^1 + (1-\delta)X_t^2 dt}(1 - \epsilon \sum_{i=1}(1-\delta)^{i-1}R_1^i(m,T) \\
&\quad - \epsilon^2 \sum_{i=1}(1-\delta)^{i-1}R_2^i(m,T) + \frac{\epsilon^2}{2}(\sum_{i=1}(1-\delta)^{i-1}R_1^i(m,T))^2 \\
&\quad + \frac{\epsilon^2}{2}\sum_{j=1}\int_m^T(\sum_{i=1}(1-\delta)^{i-1}r_{ij}^A(T,t,t))^2 dt) + o_Q(\epsilon^2) .
\end{aligned} \tag{2.11}$$

The current value of defaultable bonds with the RMV recovery rules of Example 2.1-i) is given by

$$
\begin{aligned}
p^{\epsilon}(0,T) &= e^{-\int_0^T X_t^1 dt}\{1+\frac{\epsilon^2}{2}\sum_{j=1}\int_m^T r_{1j}^A(T,t,t)^2 dt\} + o(\epsilon^2)\ , & (2.12)\\
u_{\delta}^{\epsilon}(0,T) &= \tilde{u}_{\delta}^{\epsilon}(0,T)1_{\{\tau>0\}} = e^{-\int_0^T X_t^1+(1-\delta)X_t^2 dt}\times \\
&\quad \times\{1+\frac{\epsilon^2}{2}\int_0^T\sum_{j=1}(\sum_{i=1}(1-\delta)^{i-1}r_{ij}^A(T,t,t))^2 dt + o(\epsilon^2)\} & (2.13)
\end{aligned}
$$

It is possible to calculate further the quantities of the second term in the right hand side of (2.12) and (2.13), under the volatility functional forms of Example 2.2 and Example 2.3.

Example2.2 (continued) In the affine structure framework, the integrand of (2.13) can be rewritten as

$$
\begin{aligned}
\sum_{j=1}(\sum_{i=1}(1-\delta)^{i-1}r_{ij}^A(T,t,t))^2 &= (\int_t^T Y_u^1 du)^2 (Y_t^1)^{-2}(H_{11}(t)+K_{11}^1(t)x+K_{11}^2(t)y)\\
&+2(1-\delta)(\int_t^T Y_u^1 du)(Y_t^1)^{-1}(\int_t^T Y_u^2 du)(Y_t^2)^{-1}(H_{12}(t)+K_{12}^1(t)x+K_{12}^2(t)y)\\
&+(1-\delta)^2(\int_t^T Y_u^2 du)^2(Y_t^2)^{-2}(H_{22}(t)+K_{22}^1(t)x+K_{22}^2(t)y)\ .
\end{aligned}
$$

Example2.3 (continued) In the framework in Example 2.3, the integrand of (2.13) can also be rewritten as

$$
\begin{aligned}
\sum_{j=1}(\sum_{i=1}(1-\delta)^{i-1}r_{ij}^A(T,t,t))^2 &= (\int_t^T Y_u^1 du)^2(Y_t^1)^{-2}\sigma_1(x)^2\\
&+2\rho(1-\delta)(\int_t^T Y_u^1 du)(Y_t^1)^{-1}(\int_t^T Y_u^2 du)(Y_t^2)^{-1}\sigma_1(x)\sigma_2(y)\\
&+(1-\delta)^2(\int_t^T Y_u^2 du)^2(Y_t^2)^{-2}\sigma_2(y)^2\ .
\end{aligned}
$$

It is also possible to derive the defaultable bond price with the RT recovery rules of Assumption 2.1-ii). In Section 4, the RT recovery rules will be assumed and the pricing problems of European options on defaultable bonds will be considered.

Example 2.1(iii) (continued) If the issuer of bonds is solvent at the future time m, the defaultable bond price under the RT recovery rules is given by

$$
\begin{aligned}
\tilde{w}_{\delta}^{\epsilon}(m,T) &= L-\epsilon\sum_{i=1}L_iR_1^i(m,T)-\epsilon^2\sum_{i=1}L_iR_2^i(m,T)+\frac{\epsilon^2}{2}\sum_{i,j=1}L_{ij}R_1^i(m,T)R_1^j(m,T)\\
&+\frac{\epsilon^2}{2}\sum_{i,j,k=1}\int_m^T L_{ij}r_{ik}^A(T,t,t)r_{jk}^A(T,t,t)dt & (2.14)
\end{aligned}
$$

where

$$L = L_1 = L_{11} = \delta e^{-\int_m^T X_t^1 dt} + (1-\delta)e^{-\int_m^T X_t^1 + X_t^2 dt}$$
$$L_2 = L_{21} = L_{12} = L_{22} = (1-\delta)e^{-\int_m^T X_t^1 + X_t^2 dt}.$$

3 Pricing default swaps

A default swap is a contract made between two parties to transfer credit risks. One party, who is a holder of defaultable bonds, pays a fixed premium to another party. If the issuer of defaultable bonds goes bankrupt before the maturity date of the swap contract, the party who receives a fixed premium has to guarantee the loss of bonds caused by bankruptcy. The value of the swap contract will be derived by using the extended model of Davis and Mavroidis (1997) and Nakagawa (1999). As underlying defaultable securities, coupon bonds with the RMV recovery rules in Assumption 2.1-i) will be exploited. The price of coupon bonds with the RMV recovery rules are given in Example 2.1-ii). The price of coupon bonds with the RMV recovery rules recovery rules are given in Example 2.1-ii). First, the framework of the default swap contracts with the maturity date, $\tilde{T}$, will be defined.

Definition 3.1 *i)If the default of the issuer of bonds occurs before $\tilde{T}$, the contract is stopped.* ($\tilde{T} \le T$)
ii)A holder of bonds, "A", has to pay a fixed premium c_i to the other party, "B" , at each fixed time t_i, $(i = 1, \ldots, m,\ 0 \le t_1 < \cdots < t_m \le \tilde{T})$. (The fixed side)
iii)If the issuer of bonds has gone into bankruptcy, "B" pays the differential between the notational (set at 1) and some recovered amount of defaultable bonds to "A". Here, the coupon bond model in Example 2.1-iii) is used the model of defaultable bonds. (The recovery side)

The value of the fixed side.
The current value of the default swap contracts for the fixed side is defined by

$$\begin{aligned} P_F &= E^Q[\sum_{i=1}^{m} c_i \exp(-\int_0^{t_i} r_u^\epsilon du) 1_{\{t_i \le \tau\}}] \\ &= \sum_{i=1}^{m} c_i u^\epsilon(0, t_i) \end{aligned} \tag{3.15}$$

Each term in the right hand side is calculated in (2.13).

The value of the recovery side.
The current value of the default swap contracts for the recovery side is defined by

$$\begin{aligned} P_R &= E^Q[e^{-\int_0^\tau r_s^\epsilon ds}(1 - \delta \tilde{v}_\delta^\epsilon(\tau-, T)) 1_{\{\tau \le \tilde{T}\}}] \\ &= E^Q[\int_0^{\tilde{T}} e^{-\int_0^t r_s^\epsilon + h_s^\epsilon ds} h_t^\epsilon (1 - \delta \tilde{v}_\delta^\epsilon(t, T)) dt] \end{aligned} \tag{3.16}$$

Lemma 2.1 was used to prove the equation (3.16) and it is necessary to calculate the following quantities, termed P_R^1 and P_R^2:

$$\begin{aligned} P_R^1 &= E^Q[\int_0^{\tilde{T}} e^{-\int_0^t r_s^\epsilon + h_s^\epsilon ds} h_t^\epsilon dt] \\ P_R^2 &= E^Q[\int_0^{\tilde{T}} e^{-\int_0^t r_s^\epsilon + h_s^\epsilon ds} h_t^\epsilon \tilde{v}_\delta^\epsilon(t,T) dt] \ . \end{aligned}$$

The following values should be estimated next.

$$\begin{aligned} \int_0^t r_s^\epsilon ds &= \int_0^t X_s^1 ds + \epsilon \int_0^t A_s^1 ds + \epsilon^2 \int_0^t B_s^1 ds + o_Q(\epsilon^2) \\ \int_0^t h_s^\epsilon ds &= \int_0^t X_s^2 ds + \epsilon \int_0^t A_s^2 ds + \epsilon^2 \int_0^t B_s^2 ds + o_Q(\epsilon^2) \ . \end{aligned}$$

Fubini's theorem leads to the explicit formula for the first and second order terms of the stochastic expansions as

$$\int_0^t A_s^i ds = \sum_{j=1} \int_0^t r_{ij}^A(t,s,s) dW_s^j \quad (i=1,2) \tag{3.17}$$

$$\int_0^t B_s^i ds = \sum_{j,k,l=1} \int_0^t r_{ijl}^B(t,s,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j \quad (i=1,2) \ . \tag{3.18}$$

Note that the second order terms of the stochastic expansion (3.18) have been calculated, but are not used in this section. Both equations (3.17) and (3.18) will lead to the asymptotic approximation formula for credit options in the next section. P_R^1 can be calculated as

$$\begin{aligned} P_R^1 &= E^Q[\int_0^{\tilde{T}} (X_t^2 + \epsilon A_t^2 + \epsilon^2 B_t^2) e^{-\sum_{i=1}^2 (\int_0^t X_s^i + \epsilon A_s^i + \epsilon^2 B_s^i ds)} dt] + o(\epsilon^2) \\ &= \int_0^{\tilde{T}} e^{-\int_0^t X_s^1 + X_s^2 ds} X_t^2 dt - \epsilon^2 \int_0^{\tilde{T}} e^{-\int_0^t X_s^1 + X_s^2 ds} (\int_0^t \sum_{i,j=1} \sigma_{2j}^A(t,u) r_{ij}^A(t,u,u) du) dt \\ &\quad + \frac{\epsilon^2}{2} \int_0^{\tilde{T}} e^{-\int_0^t X_s^1 + X_s^2 ds} X_t^2 (\int_0^t \sum_{j=1} (\sum_{i=1} r_{ij}^A(t,u,u))^2 du) dt + o(\epsilon^2) \ . \end{aligned} \tag{3.19}$$

The quantity $\sum_{i,j=1}^2 (\sigma_{2j}^A(t,u) r_{ij}^A(t,u,u))$ which appears as the second term of the right hand side of (3.19) can be calculated further using under the setting of Example 2.2 and 2.3. The remaining task which must be accomplished is the evaluation of P_R^2. For that purpose, new variables $S_1^{ij}(t,T)$ $(i,j=1,2)$ and $S_1^{ijk}(t,T)$ $(i,j,k=1,2)$ are introduced as

$$\begin{aligned} S_1^{ij}(t,T) &= \int_0^t s_{ij}^A(T,s) dW_s^j \quad (i,j=1,2) \\ S_2^{ijk}(t,T) &= \sum_{l=1} \int_0^t s_{ijl}^B(T,s) \int_0^s \sigma_{lk}^A(s,u) dW_u^k dW_s^j \quad (i,j=1,2) \end{aligned}$$

where

$$
\begin{aligned}
s_{1j}^A(T,t,s) &= r_{1j}^A(t,s,s) + r_{1j}^A(T,t,s) = r_{1j}^A(T,s,s)\\
s_{2j}^A(T,t,s) &= r_{2j}^A(t,s,s) + (1-\delta) r_{2j}^A(T,t,s)\\
&= (\int_s^t Y_u^2 du + (1-\delta)\int_t^T Y_u^2 du)(Y_s^2)^{-1}\sigma_{2j}(X_s^1, X_s^2)\\
s_{1jk}^B(T,t,s) &= r_{1jk}^B(t,s,s) + r_{1jk}^B(T,t,s) = r_{1jk}^B(T,s,s)\\
s_{2jk}^B(T,t,s) &= r_{2jk}^B(t,s,s) + (1-\delta) r_{2jk}^B(T,t,s)\\
&= (\int_s^t Y_u^2 du + (1-\delta)\int_t^T Y_u^2 du)(Y_s^2)^{-1}\partial_k\sigma_{2j}(X_s^1, X_s^2)\ .
\end{aligned}
$$

Note that $S_1^{ij}(t,T)$ and $S_2^{ijk}(t,T)$ have relations

$$
S_1^{ij}(t,T) = \int_0^t A_s^{ij} ds + R_1^{ij}(t,T) \text{ and } S_2^{ijk}(t,T) = \int_0^t B_s^{ijk} ds + R_2^{ijk}(t,T)\ ,
$$

with A_t^{ij} and B_t^{ijk} given at (2.6) and (2.7). The quantity of P_R^2 can be calculated as

$$
\begin{aligned}
P_R^2 &= \sum_{g=1} b_g E^Q[\int_0^{u_g\wedge\tilde{T}} e^{-\int_0^t r_s^\epsilon + h_s^\epsilon ds} h_t^\epsilon \tilde{u}_\delta^\epsilon(t,u_g)dt]\\
&= \sum_{g=1} b_g E^Q[\int_0^{u_g\wedge\tilde{T}} \{e^{-\int_0^{u_g} X_s^1 + X_s^2 ds + \delta\int_t^{u_g} X_s^2 ds} X_t^2\}\{1 + \epsilon(\frac{A_t^2}{X_t^2} - \sum_{i=1} S_1^i(t,u_g))\\
&\quad + \epsilon^2(\frac{B_t^2}{X_t^2} - \sum_{i=1} S_2^i(t,u_g)) + \frac{\epsilon^2}{2}(\sum_{i=1} S_1^i(t,u_g))^2 - \epsilon^2\frac{A_t^2}{X_t^2}(\sum_{i=1} S_1^i(t,u_g))\\
&\quad + \frac{\epsilon^2}{2}\int_t^{u_g}\sum_{j=1}(\sum_{i=1}(1-\delta)^{i-1} r_{ij}^A(t,u_g))^2 ds)\}dt] + o(\epsilon^2)\\
&= \sum_{g=1} b_g \int_0^{u_g\wedge\tilde{T}} e^{-\int_0^{u_g} X_s^1 + X_s^2 ds + \delta\int_t^{u_g} X_s^2 ds}\{X_t^2 + \frac{\epsilon^2}{2}X_t^2\int_0^t\sum_{j=1}(\sum_{i=1} s_{ij}^A(u_g,t,s))^2 ds\\
&\quad - \epsilon^2\int_0^t\sum_{i,j=1}\sigma_{2j}^A(t,s)s_{ij}^A(u_g,t,s)ds\\
&\quad + \frac{\epsilon^2}{2}\int_t^{u_g} X_t^2\sum_{j=1}(\sum_{i=1}(1-\delta)^{i-1} r_{ij}^A(u_g,s,s))^2 ds\}dt + o(\epsilon^2)
\end{aligned}
$$

It is also possible to calculate further the terms which appear as the second terms of the right hand side, $\sum_{j=1}^2(\sum_{i=1}^2 s_{ij}^A(u_g,t,s))^2$ $(g = 1,\cdots,n)$, under the setting of Example 2.2 and 2.3. Numerical studies of the default swap price of the recovery side under the models with the affine structures will be demonstrated in Section 5.

4 Pricing options on defaultable bonds

Despite of their importance, it seems that there has been little attention given to the pricing problems of options on defaultable bonds. See Jarrow and Turnbull (1995) and

Muroi (2002) for related topics. Jarrow and Turnbull (1995) used as a framework the forward rate process of Heath, Jarrow and Morton (1992). In Muroi (2002), the PDE approach was exploited for the pricing problem of American put options on defaultable bonds. In this paper, the value of European put options on defaultable bonds will be derived by the asymptotic expansion approach, using the RT recovery rules under the setting of Example 2.1-iii) to price the underlying defaultable securities. In order to evaluate default put options by the asymptotic expansion approach, the option premium will be divided into three blocks which will be called the main part, the bond option part, and the remained part, respectively. The price of European put options on defaultable bonds at time, t, with the strike price, K, and maturity, m, on defaultable bonds with maturity, T, will be denoted by $P_K(t,m)$. The current price of European put options on defualtable bonds is given by

$$\begin{aligned} P_K(0,m) &= E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - w_\delta^\epsilon(m,T))^+] \\ &= E^Q[e^{-\int_0^m r_t^\epsilon dt}\{(K - \tilde{w}_\delta^\epsilon(m,T))^+ 1_{\{\tau>m\}} + (K - \delta p^\epsilon(m,T))^+ 1_{\{\tau\le m\}}\}] \\ &= E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - \tilde{w}_\delta^\epsilon(m,T))^+ 1_{\{\tau>m\}}] + E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - \delta p^\epsilon(m,T))^+] \\ &\quad - E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - \delta p^\epsilon(m,T))^+ 1_{\{\tau>m\}}] \\ &= E^Q[e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - \tilde{w}_\delta^\epsilon(m,T))^+] + E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - \delta p^\epsilon(m,T))^+] \\ &\quad - E^Q[e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - \delta p^\epsilon(m,T))^+] \,. \end{aligned} \tag{4.20}$$

The last equality in (4.20) is due to Lemma 2.1. Each term in (4.20) will be named and new random variables, d_i^ϵ $(i = 1,2,3)$ will be introduced as below.

i)The first part, $P_K^1(0,m) = E^Q[e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - \tilde{w}_\delta^\epsilon(m,T))^+]$, will be termed "the main part" with a new random variable, $d_1^\epsilon = e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - \tilde{w}_\delta^\epsilon(m,T))$, introduced.

ii)The second part, $P_K^2(0,m) = E^Q[e^{-\int_0^m r_t^\epsilon dt}(K - p^\epsilon(m,T))^+]$, will be termed "the bond option part", and a the new random variable, $d_2^\epsilon = e^{-\int_0^m r_t^\epsilon dt}(K - p^\epsilon(m,T))$, introduced.

iii)The third part, $P_K^3(0,m) = E^Q[e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - p^\epsilon(m,T))^+]$ will be termed "the remained part", again with a new random variable, $d_3^\epsilon = e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - p^\epsilon(m,T))$.

The default option premium $P_K(0,m)$ will be calculated as

$$\begin{aligned} P_K(0,m) &= P_K^1(0,m) + \delta P_{K/\delta}^2(0,m) - \delta P_{K/\delta}^3(0,m) \ (\delta \neq 0) \\ P_K(0,m) &= P_K^1(0,m) + K p^\epsilon(0,m) - K u^\epsilon(0,m) \ (\delta = 0) \end{aligned}$$

If the fractional recovery rate δ equals 0, it is possible to obtain the numerical approximation formula for the price of default put options by using (2.12), (2.13), and the asymptotic expansion formula of the main part. The stochastic expansion of three terms will be calculated as below.

i)The main part

The main part of the option premium, d_1^ϵ, will be expanded as

$$\begin{aligned}
d_1^\epsilon &= e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - \tilde{w}_\delta^\epsilon(m,T)) \\
&= e^{-\sum_{i=1}^2 \int_0^m X_t^i dt}\{1 - \sum_{i=1}\epsilon\int_0^m A_t^i dt - \sum_{i=1}\epsilon^2\int_0^m B_t^i dt + \frac{\epsilon^2}{2}(\sum_{i=1}\int_0^m A_t^i dt)^2 + o_Q(\epsilon^2)\} \\
&\quad \times\{K - L + \epsilon\sum_{i=1} L_i R_1^i(m,T) + \epsilon^2\sum_{i=1} L_i R_2^i(m,T) - \frac{\epsilon^2}{2}\sum_{i,j=1} L_{ij}R_1^i(m,T)R_1^j(m,T) \\
&\quad -\frac{\epsilon^2}{2}\sum_{i,j,k=1}\int_m^T L_{ij} r_{ik}^A(T,t,t) r_{jk}^A(T,t,t)dt + o_Q(\epsilon^2)\} \\
&= g_0^1 + \epsilon g_1^1 + \epsilon^2 g_2^1 + o_Q(\epsilon^2)\ .
\end{aligned}$$

where R_1^i and R_2^i are given at (2.8) and (2.9). This calculation leads to the zeroth, first and second order terms of the stochastic expansion of the main part, g_0^1, g_1^1, and g_2^1, which are given by

$$\begin{aligned}
g_0^1 &= e^{-\sum_{i=1}^2 \int_0^m X_t^i dt}(K - L) \\
g_1^1 &= -g_0^1\sum_{i=1}\int_0^m A_t^i dt + e^{-\sum_{i=1}^2 \int_0^m X_t^i dt}\sum_{i=1} L_i R_1^i(m,T) \\
&= \sum_{j=1}\int_0^m \sigma_j^1(t)dW_t^j \\
g_2^1 &= -g_0^1\{\sum_{i=1}\int_0^m B_t^i dt - \frac{1}{2}(\sum_{i=1}\int_0^m A_t^i dt)^2\} + e^{-\sum_{i=1}^2 \int_0^m X_t^i dt}\{\sum_{i=1} L_i R_2^i(m,T) \\
&\quad -\frac{1}{2}\sum_{i,j=1} L_{ij}R_1^i(m,T)R_1^j(m,T) - \frac{1}{2}\sum_{i,j,k=1}\int_m^T L_{ij} r_{ik}^A(T,t,t) r_{jk}^A(T,t,t)dt \\
&\quad -(\sum_{i=1}\int_0^m A_t^i dt)(\sum_{i=1} L_i R_1^i(m,T))\}
\end{aligned}$$

where $\sigma_j^1(t) = \sum_{i=1}^2(-g_0^1 r_{ij}^A(m,t,t) + e^{-\sum_{k=1}^2 \int_0^m X_t^k dt} L_i r_{ij}^A(T,m,t))$. The explicit formulas of $\sigma_j^1(t) \quad (j = 1, 2)$ were derived by using (3.17) in Section 3.

ii)The bond option part

The bond option part, d_2^ϵ, will be expanded as

$$\begin{aligned}
d_2^\epsilon &= e^{-\int_0^m r_t^\epsilon dt}(K - p^\epsilon(m,T)) \\
&= e^{-\int_0^m X_t^1 dt}\{1 - \epsilon\int_0^m A_t^1 dt - \epsilon^2\int_0^m B_t^1 dt + \frac{\epsilon^2}{2}(\int_0^m A_t^1 dt)^2 + o_Q(\epsilon^2)\}
\end{aligned}$$

$$\times\{K - e^{-\int_m^T X_t^1 dt}(1 - \epsilon R_1^1(m,T) - \epsilon^2 R_2^1(m,T) + \frac{\epsilon^2}{2}(R_1^1(m,T))^2$$
$$+\frac{\epsilon^2}{2}\sum_{j=1}\int_m^T r_{1j}^A(T,t,t)^2 dt + o_Q(\epsilon^2)\}$$
$$= g_0^2 + \epsilon g_1^2 + \epsilon^2 g_2^2 + o_Q(\epsilon^2)\ .$$

This calculation leads to the zeroth, first and second order terms of the stochastic expansion of the bond option part, g_0^2, g_1^2 and g_2^2, given by

$$\begin{aligned}
g_0^2 &= Ke^{-\int_0^m X_t^1 dt} - e^{-\int_0^T X_t^1 dt} \\
g_1^2 &= -g_0^2\int_0^m A_t^1 dt + e^{-\int_0^T X_t^1 dt} R_1^1(m,T) \\
&= \sum_{j=1}\int_0^m \sigma_j^2(t) dW_t^j \\
g_2^2 &= -g_0^2(\int_0^m B_t^1 dt - \frac{1}{2}(\int_0^m A_t^1 dt)^2) + e^{-\int_0^T X_t^1 dt}\{R_2^1(m,T) - \frac{1}{2}(R_1^1(m,T))^2 \\
&\quad -\frac{1}{2}\sum_{j=1}\int_m^T r_{1j}^A(t,t)^2 dt - R_1^1(m,T)\int_0^m A_t^1 dt\}
\end{aligned}$$

where $\sigma_j^2(t) = -g_0^2 r_{1j}^A(m,t,t) + e^{-\int_0^T X_t^1 dt} r_{1j}^A(T,m,t)$. The explicit formulas of $\sigma_j^2(t)$ $(j = 1,2)$ were also derived by using (3.17) in Section 3.

iii)The remained part

The remained part, d_3^ϵ, will be expanded as

$$\begin{aligned}
d_3^\epsilon &= e^{-\int_0^m r_t^\epsilon + h_t^\epsilon dt}(K - p^\epsilon(m,T)) \\
&= e^{-\sum_{i=1}^2\int_0^m X_t^i dt}\{1 - \sum_{i=1}\epsilon\int_0^m A_t^i dt - \sum_{i=1}\epsilon^2\int_0^m B_t^i dt + \frac{\epsilon^2}{2}(\sum_{i=1}\int_0^m A_t^i dt)^2 + o_Q(\epsilon^2)\} \\
&\quad \times\{K - e^{-\int_m^T X_t^1 dt}(1 - \epsilon R_1^1(m,T) - \epsilon^2 R_2^1(m,T) + \frac{\epsilon^2}{2}(R_1^1(m,T))^2 \\
&\quad +\frac{\epsilon^2}{2}\sum_{j=1}\int_m^T r_{1j}^A(T,t,t)^2 dt + o_Q(\epsilon^2)\} \\
&= g_0^3 + \epsilon g_1^3 + \epsilon^2 g_2^3 + o_Q(\epsilon^2)\ .
\end{aligned}$$

This calculation leads to the zeroth, first and second order terms of the stochastic expansion of the bond option part, g_0^3, g_1^3 and g_2^3, given by

$$\begin{aligned}
g_0^3 &= Ke^{-\sum_{i=1}^2\int_0^m X_t^i dt} - e^{-\int_0^T X_t^1 dt} e^{-\int_0^m X_t^2 dt} \\
g_1^3 &= -g_0^3\sum_{i=1}\int_0^m A_t^i dt + e^{-\int_0^T X_t^1 dt} e^{-\int_0^m X_t^2 dt} R_1^1(m,T) \\
&= \sum_{j=1}\int_0^m \sigma_j^3(t) dW_t^j
\end{aligned}$$

$$g_2^3 = -g_0^3(\sum_{i=1}^{2}\int_0^m B_t^i dt - \frac{1}{2}(\sum_{i=1}^{2}\int_0^m A_t^i dt)^2) + e^{-\int_0^T X_t^1 dt} e^{-\int_0^m X_t^2 dt}\{(R_2^1(m,T)$$
$$-\frac{1}{2}(R_1^1(m,T))^2 - \frac{1}{2}\sum_{j=1}^{2}\int_m^T r_{1j}^A(T,t,t)^2 dt - R_1^1(m,T)\sum_{i=1}^{2}\int_0^m A_t^i dt\}$$

where $\sigma_j^3(t) = (-\sum_{i=1}^{2} g_0^3 r_{ij}^A(m,t,t)) + e^{-\int_0^T X_t^1 dt} e^{-\int_0^m X_t^2 dt} r_{1j}^A(T,m,t)$. The explicit formulas of $\sigma_j^3(t)$ $(j = 1,2)$ were also derived by using (3.17) in Section 3.

It is possible to calculate the second terms of the stochastic expansions, g_2^i $(i = 1,2,3)$, further by using (3.17) and (3.18) in Section 3, and this calculation will be used to derive the asymptotic expansions of the density functions for the random variables, X_i^ϵ, later. The characteristic functions of the random variables, $X_i^\epsilon = \epsilon^{-1}(d_i^\epsilon - g_0^i)$ $(i = 1,2,3)$, which are denoted as $\psi_i(t)$ $(i = 1,2,3)$, will be calculated next. Once the asymptotic expansions of the density functions, $f_i^\epsilon(x)$ $(i = 1,2,3)$, of the random variable, X_i^ϵ, have been solved, it is possible to derive the asymptotic approximation formula of the credit option premiums. Notice that the random variables with a small parameter, ϵ, $X_i^\epsilon = g_1^i + \epsilon g_2^i + o_Q(\epsilon)$ $(i = 1,2,3)$ converge to g_1^i $(i = 1,2,3)$ as $\epsilon \to 0$. The random variables g_1^i $(i = 1,2,3)$ follow non-degenerate Gaussian distributions. The distribution functions of the random variables, X_t^i $(i = 1,2,3)$, are expected to be represented as the limiting Gaussian distribution functions plus some adjustment terms with smaller orders with respect to the parameter, ϵ, when ϵ is sufficiently small. The formal asymptotic expansions of the characteristic functions, $\psi_i(t)$ $(i = 1,2,3)$, will be calculated as

$$\psi_i(t) = E^Q[e^{itX_i^\epsilon}] = E^Q[e^{itg_1^i}(1 + \epsilon it E^Q[g_2^i|g_1^i])] + o(\epsilon) . \quad (4.21)$$

It is possible to show that the conditional expectations $E^Q[g_2^i|g_1^i = x]$ $(i = 1,2,3)$ are represented by some polynomial functions of x by Lemma 4.1. This lemma is a slight generalization of Lemma 5.7 of Yoshida (1992).

Lemma 4.1 *Let $\{\mathbf{w}_t\}$ be a n-dimensional standard Brownian motion and $\mathbf{x}$ be a k dimensional vector. Suppose there exists a non-stochastic function, $\mathbf{q}_1(t) : \mathbf{R}^1 \to \mathbf{R}^{k\times n}$, and a positive definite matrix, $\mathbf{\Sigma} = \int_0^T \mathbf{q}_1(t)\mathbf{q}_1(t)'dt$.*
(1)*Suppose there exist non-stochastic functions $\mathbf{q}_i(t) : \mathbf{R}^1 \to \mathbf{R}^{m\times n}$ $(i = 2,3)$, then for $0 \le s \le t \le T$*

$$E[\int_0^t [\int_0^s \mathbf{q}_2(u)d\mathbf{w}_u]'\mathbf{q}_3(s)d\mathbf{w}_s | \int_0^T \mathbf{q}_1(u)d\mathbf{w}_u = \mathbf{x}]$$
$$= \mathbf{tr}\int_0^t [\int_0^s \mathbf{q}_2(u)\mathbf{q}_1(u)'du]\mathbf{q}_1(s)\mathbf{q}_3(s)'ds\mathbf{\Sigma}^{-1}[\mathbf{x}\mathbf{x}' - \mathbf{\Sigma}]\mathbf{\Sigma}^{-1} .$$

(2)*Suppose there exist non-stochastic functions $\mathbf{q}_i(t) : \mathbf{R}^1 \to \mathbf{R}^n$ $(i = 2,3)$, then for $0 \le s \le t \le T$*

$$E[\int_0^s \mathbf{q}_2(u)d\mathbf{w}_u][\int_0^t \mathbf{q}_3(v)d\mathbf{w}_v] | \int_0^T \mathbf{q}_1(u)d\mathbf{w}_u = \mathbf{x}]$$

$$= \int_0^s \mathbf{q}_2(u)\mathbf{q}_3(u)'du + [\int_0^s \mathbf{q}_2(u)\mathbf{q}_1(u)'du]\mathbf{\Sigma}^{-1}[\mathbf{x}\mathbf{x}' - \mathbf{\Sigma}]\mathbf{\Sigma}^{-1}[\int_0^t \mathbf{q}_1(u)\mathbf{q}_3(u)'du] .$$

As the next step, the following lemma which was given by Fujikoshi, Morimune, Kunitomo and Taniguchi (1982) will be used to invert the characteristic functions, $\psi_i(t)$ $(i = 1, 2, 3)$ of (4.21).

Lemma 4.2 *The random variable* $\mathbf{x}$ *follows an* n*-dimensional normal distribution with the mean* $\mathbf{0}$ *and the variance-covariance matrix* $\mathbf{\Sigma}$*. Then, for any polynomial functions* $g(\cdot)$ *and* $h(\cdot)$*, the Fourier inversion formula of* $h(-i\mathbf{t})E[g(\mathbf{x})e^{i\mathbf{t}'\mathbf{x}}]$ *is given by*

$$\mathcal{F}^{-1}[h(-i\mathbf{t})E[g(\mathbf{x})e^{i\mathbf{t}'\mathbf{x}}]]_{\langle\xi\rangle} = h(\frac{\partial}{\partial\xi})g(\xi)\phi_\Sigma(\xi)$$

where $\mathcal{F}^{-1}_{\langle\xi\rangle}$ *is the* n *dimensional Fourier inversion being evaluated at* $\xi \in \mathbf{R}^n$*,*

$$\mathcal{F}^{-1}[h(-i\mathbf{t})E[g(\mathbf{x})e^{i\mathbf{t}'\mathbf{x}}\mathbf{x}]]_{\langle\xi\rangle} = (\frac{1}{2\pi})^n \int_{\mathbf{R}^n} e^{-i\mathbf{t}'\xi}h(-i\mathbf{t})E[g(\mathbf{x})e^{i\mathbf{t}'\mathbf{x}}]d\mathbf{t} .$$

This leads to Lemma 4.3 of the asymptotic expansion formula of the density functions of the random variables, X_i^ϵ $(i = 1, 2, 3)$, as described by Kunitomo and Takahashi (2001).

Lemma 4.3 *The asymptotic expansions of the density functions of the random variables,* $X_i^\epsilon(i = 1, 2, 3)$*, are given by*

$$f_i^\epsilon(x) = \phi_{\Sigma_i}(x) + \epsilon[\frac{c_i}{\Sigma_i}x^3 + (\frac{f_i}{\Sigma_i} - 2c_i)x]\phi_{\Sigma_i}(x) + O(\epsilon^2) ,$$

where $\phi_\Sigma(x)$ *stands for the Gaussian density function with mean* 0 *and variance* Σ*. The constants* c_i *and* f_i *are determined by*

$$E[g_2^i | g_1^i = x] = c_i x^2 + f_i \quad (i = 1, 2, 3) .$$

It is possible to derive the asymptotic expansions of the random variables, X_i^ϵ, by deriving the constants c_i, f_i in the previous lemma $(i = 1, 2, 3)$.

i)The main part

The coefficients c_1 and f_1 in Lemma 4.3 will be derived by Lemma 4.1, and by (3.17) and (3.18) in Section 3.

$$\begin{aligned} c_1 &= -\frac{g_0^1}{(\Sigma_1)^2}\{\sum_{i,j,k,l=1} \int_0^m r_{ijl}^B(m,t,t)\sigma_j^1(t)\int_0^t \sigma_{lk}^A(t,s)\sigma_k^1(s)dsdt \\ &\quad -\frac{1}{2}(\sum_{i,j=1} \int_0^m r_{ij}^A(m,t,t)\sigma_j^1(t)dt)^2\} \\ &\quad +\frac{e^{-\sum_{i=1}^2 \int_0^m X_t^i dt}}{(\Sigma_1)^2}\{\sum_{i,j,k,l=1} L_i \int_0^m r_{ijl}^B(T,m,t)\sigma_j^1(t)\int_0^t \sigma_{lk}^A(t,s)\sigma_k^1(s)dsdt \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{i,j=1} L_{ij}(\sum_{k=1}\int_0^m r^A_{ik}(T,m,t)\sigma^1_k(t)dt)(\sum_{k=1}\int_0^m r^A_{jk}(T,m,t)\sigma^1_k(t)dt) \\
&-(\sum_{i,j=1}\int_0^m r^A_{ij}(m,t,t)\sigma^1_j(t)dt)(\sum_{i,j=1} L_i\int_0^m r^A_{ij}(T,m,t)\sigma^1_j(t)dt)\} \ , \\
b_1 \ &= \ \frac{g^1_0}{2}\sum_{j=1}\int_0^m(\sum_{i=1} r^A_{ij}(m,t,t))^2 dt \\
&-e^{-\sum_{i=1}^2\int_0^m X^i_t dt}\{\frac{1}{2}\sum_{i,j,k=1} L_{ij}\int_0^m r^A_{ik}(T,m,t)r^A_{jk}(T,m,t)dt \\
&+\frac{1}{2}\sum_{i,j,k=1}\int_m^T L_{ij}r^A_{ik}(T,t,t)r^A_{jk}(T,t,t)dt+\sum_{i,j,k=1}\int_0^m L_j r^A_{ik}(m,t,t)r^A_{jk}(T,m,t)dt\} \ ,
\end{aligned}
$$

and

$$f_1 = -c_1\Sigma_1 + b_1 \ ,$$

where $\Sigma_1 = \int_0^m(\sigma^1_1(t))^2 + (\sigma^2_1(t))^2 dt$.

ii)The bond option part

The coefficients c_2 and f_2 in Lemma 4.3 will be also derived by Lemma 4.1, and by (3.17) and (3.18) in Section 3.

$$
\begin{aligned}
c_2 \ &= \ -\frac{g^2_0}{(\Sigma_2)^2}\{\sum_{j,k,l=1}\int_0^m r^B_{1jl}(m,t,t)\sigma^2_j(t)\int_0^t \sigma^A_{lk}(t,s)\sigma^2_k(s)dsdt \\
&-\frac{1}{2}(\sum_{j=1}\int_0^m r^A_{1j}(m,t,t)\sigma^2_j(t)dt)^2\} \\
&+\frac{e^{-\int_0^T X^1_t dt}}{(\Sigma_2)^2}\{\sum_{j,k,l=1}\int_0^m r^B_{1jl}(T,m,t)\sigma^2_j(t)\int_0^t \sigma^A_{lk}(t,s)\sigma^2_k(s)dsdt \\
&-\frac{1}{2}(\sum_{j=1}\int_0^m r^A_{1j}(T,m,t)\sigma^2_j(t)dt)^2 \\
&-(\sum_{j=1}\int_0^m r^A_{1j}(m,t,t)\sigma^2_j(t)dt)(\sum_{j=1}\int_0^m r^A_{1j}(T,m,t)\sigma^2_j(t)dt)\} \ , \\
b_2 \ &= \ \frac{g^2_0}{2}\sum_{j=1}\int_0^m r^A_{1j}(m,t,t)^2 dt - e^{-\int_0^T X^1_t dt}\{\frac{1}{2}\sum_{j=1}\int_0^m r^A_{1j}(T,m,t)^2 dt \\
&+\frac{1}{2}\sum_{j=1}\int_m^T r^A_{1j}(T,t,t)^2 dt + \sum_{j=1}\int_0^m r^A_{1j}(m,t,t)r^A_{1j}(T,m,t)dt\} \ ,
\end{aligned}
$$

and

$$f_2 = -c_2\Sigma_2 + b_2 \ ,$$

where $\Sigma_2 = \int_0^m(\sigma^1_2(t))^2 + (\sigma^2_2(t))^2 dt$.

iii)The remained part

Derivation of the coefficients c_3 and f_3 in Lemma 4.3 will be the same as in the main and bond option parts.

$$\begin{aligned}
c_3 &= -\frac{g_0^3}{(\Sigma_3)^2}\{\sum_{i,j,k,l=1}\int_0^m r_{ijl}^B(m,t,t)\sigma_j^3(t)\int_0^t \sigma_{lk}^A(t,s)\sigma_k^3(s)dsdt \\
&\quad -\frac{1}{2}(\sum_{i,j=1}\int_0^m r_{ij}^A(m,t,t)\sigma_j^3(t)dt)^2\} \\
&\quad +\frac{e^{-\int_0^T X_t^1 dt}e^{-\int_0^m X_t^2 dt}}{(\Sigma_3)^2}\{\sum_{j,k,l=1}\int_0^m r_{1jl}^B(T,m,t)\sigma_j^3(t)\int_0^t \sigma_{lk}^A(t,s)\sigma_k^3(s)dsdt \\
&\quad -\frac{1}{2}(\sum_{j=1}\int_0^m r_{1j}^A(T,m,t)\sigma_j^3(t)dt)^2 \\
&\quad -(\sum_{i,j=1}\int_0^m r_{ij}^A(m,t,t)\sigma_j^3(t)dt)(\sum_{j=1}\int_0^m r_{1j}^A(T,m,t)\sigma_j^3(t)dt)\}\ , \\
b_3 &= \frac{g_0^3}{2}\sum_{j=1}\int_0^m(\sum_{i=1} r_{ij}^A(m,t,t))^2dt - e^{-\int_0^T X_t^1 dt}e^{-\int_0^m X_t^2 dt}\{\frac{1}{2}\sum_{j=1}\int_0^m r_{1j}^A(T,m,t)^2dt \\
&\quad +\frac{1}{2}\sum_{j=1}\int_m^T r_{1j}^A(T,t,t)^2dt + \sum_{i,j=1}\int_0^m r_{ij}^A(m,t,t)r_{1j}^A(T,m,t)dt\}\ ,
\end{aligned}$$

and

$$f_3 = -c_3\Sigma_3 + b_3\ ,$$

where $\Sigma_3 = \int_0^m (\sigma_3^1(t))^2 + (\sigma_3^2(t))^2 dt$.

The asymptotic expansions of the density functions for the random variables, X_i^ϵ $(i = 1,2,3)$, are derived and it is possible to obtain the asymptotic approximation formula for the price of options on defaultable bonds, by calculating $P_K^i(0,m)$ as,

$$\begin{aligned}
P_K^i(0,m) &= E^Q[g_0^i + \epsilon X_i^\epsilon]^+ \\
&= \int_{g_0^i+\epsilon x\geq 0}(g_0^i+\epsilon x)f_i^\epsilon(x)dx \\
&= \int_{g_0^i+\epsilon x\geq 0}(g_0^i+\epsilon x)\phi_{\Sigma_i}(x)\{1+\epsilon[\frac{c_i}{\Sigma_i}x^3 + (\frac{f_i}{\Sigma_i}-2c_i)x] + \epsilon^2 h_i(x)dx\} + o(\epsilon^2)\ ,
\end{aligned}$$

where $\phi_\Sigma(\cdot)$ is a density function of the Gaussian distribution with the mean 0 and the variance and covariance matrix Σ. Applying the integration-by-parts formula will lead immediately to Lemma 4.4.

Lemma 4.4 *The asymptotic expansions of* $P_K^i(0,m)$ $(i = 1,2,3)$ *as* $\epsilon \to 0$ *are given by*

$$P_K^i(0,m) = g_0^i\int_{-y_\epsilon^i}^\infty \phi_{\Sigma_i}(x)dx + \epsilon\int_{-y_\epsilon^i}^\infty x\phi_{\Sigma_i}(x)dx + \epsilon^2\int_{-y_\epsilon^i}^\infty (c_i x^2 + f_i)\phi_{\Sigma_i}(x)dx + o(\epsilon^2)$$

where $y_\epsilon^i = (1/\epsilon)g_0^i$.

The following formulas which are derived by integration-by-parts formula have been used to evaluate the integral in the previous Lemma 4.4.

$$\int_{-y}^{\infty} x\phi_\Sigma(x)dx = \Sigma\phi_\Sigma(x) ,$$
$$\int_{-y}^{\infty} x^2\phi_\Sigma(x)dx = \Sigma\Phi(\frac{y}{\Sigma^{1/2}}) - y\Sigma\phi_\Sigma(y) ,$$

where $\Phi(\cdot)$ is a normal Gaussian distribution function. By using Lemma 4.4, it is possible to calculate each of the three blocks in (4.20). This calculation leads to the asymptotic approximation of the option price.

5 Numerical Results

In the previous sections, the pricing problems of credit derivatives have been examined in cases in which the dimensional stochastic processes r_t and h_t follow the diffusion processes,

$$r_t^\epsilon = x_1 + \int_0^t \mu_1(\bar{x}_1 - r_s^\epsilon)ds + \sum_{j=1} \epsilon \int_0^t \sigma_{1j}(r_s^\epsilon, h_s^\epsilon)dW_s^j \tag{5.22}$$
$$h_t^\epsilon = x_2 + \int_0^t \mu_2(\bar{x}_2 - h_s^\epsilon)ds + \sum_{j=1} \epsilon \int_0^t \sigma_{2j}(r_s^\epsilon, h_s^\epsilon)dW_s^j . \tag{5.23}$$

Some numerical examples for the pricing problem of credit derivatives in the previous sections will be given under two cases, Gaussian case and Non-Gaussian case. (i) the underlying stochastic process (5.22) and (5.23) are Gaussian processes (Gaussian case), (ii) the underlying stochastic process (5.22) and (5.23) are the affine structure model but not Gaussian processes (non-Gaussian case).

In the previous sections, the pricing problems of contingent claims with credit risk are discussed in the environment that the stochastic processes, r_t^ϵ and h_t^ϵ, are positive. However, it is not assumed that the stochastic interest rate process, r_t^ϵ, and the hazard rate process, h_t^ϵ, are positive processes in this section and the argument used here is not able to justify in the rigorous sense. But such a model is commonly used for the sake of the analytical tractability.

(i)Gaussian case. The numerical value of default swaps of the recovery side when $x_1 = 0.03$ and $x_2 = 0.05$ is shown in Table 1. The parameters of the diffusion processes (5.22) and (5.23) were chosen as $\epsilon = 0.01, \mu_1 = 0.1, \mu_2 = 0.4, \bar{x}_1 = 0.05$, and $\bar{x}_2 = 0.04$ and the volatility functions as $\sigma_{11}(r,h) = 1.0$, $\sigma_{12}(r,h) = 0.0$, $\sigma_{21}(r,h) = -1.2$, and $\sigma_{22}(r,h) = 1.6$. Further, the recovery rate was fixed at $\delta = 0.4$, and the stochastic processes r_t, h_t are Gaussian processes. The asymptotic approximation formula for the value of the default swap contracts was given in Section 3. The time to the maturity date for the swap contracts were $\tilde{T} = 2$ (years) and the time to maturity for defaultable bonds table1:Swap value

Monte Carlo	0.049833
Asymptotic Expansion	0.049885
difference rate(%)	0.104 %

table3:Swap value

Monte Carlo	0.049838
Asymptotic Expansion	0.049891
difference rate(%)	0.106 %

table2:Default option premium

Monte Carlo	0.021699
Asymptotic Expansions	
first order	0.021740
difference rate(%)	0.189 %
second order	0.021703
difference rate(%)	0.018 %

table4:Default option premium

Monte Carlo	0.021719
Asymptotic Expansions	
first order	0.021779
difference rate(%)	0.276 %
second order	0.021765
difference rate(%)	0.211 %

Table 1 and Table 2 present the Gaussian case and Table 3 and Table 4 present non-Gaussian case respectively. The asymptotic approximation value of the swap contracts were derived by using second orders of the asymptotic expansion. The asymptotic approximation value of default options were derived by using first and second orders of the asymptotic expansions.

was also $T = 2$ (years). A defaultable coupon bond pays the coupon rate of \$0.05 every half year as coupons, and the face value of \$1.0 and coupon rate of \$0.05 at the maturity date, if bankruptcy has not occurred before the maturity date. For comparative purposes, (3.16) was calculated numerically by Monte Carlo simulations, with the number of the simulations, $M = 300,000$ and the number of time steps $N = 5,000$ for 1 year. Thus, the numerical results are expected to be very accurate.

A numerical example for European put options on the defaultable bonds has been given also in Table 2, the value of which was derived by the asymptotic approximation formula in Section 4. The parameters for the diffusion processes (5.22) and (5.23) were the same as those used in credit default swaps. The time to maturity of the defaultable bonds used was $T = 3$ (years) and the time to maturity of options was $m = 1$ (year). The strike price of put options was fixed at $K = 0.85$. For comparative purposes, the value of (4.20) was calculated numerically by Monte Carlo simulations.

(ii)non-Gaussian case. The numerical studies using other volatility functions were also examined. The volatility functions are selected as $\sigma_{11}(r, h) = \sqrt{r}$, $\sigma_{12}(r, h) = 0.0$, $\sigma_{21}(r, h) = -0.6\sqrt{r}$, $\sigma_{22}(r, h) = 1.5\sqrt{0.02 + h}$ with the small parameter $\epsilon = 0.05$. Other parameters were the same as in the two previous examples. In this case, however, the stochastic processes, r_t and h_t were not Gaussian processes. It is also possible to derive the value of the swap contracts and put options as shown in Table 3, in which the value of the swap contracts was calculated by Monte Carlo simulations and by asymptotic expansion methods. The numerical value of put options in Table 4 using the asymptotic expansion approach for credit derivatives appears very accurate.

6 Concluding Remarks

The analytic approximation formulas for the value of defaulable bonds, default swaps, and European put options on defaultable bonds were derived by asymptotic expansions methods. These seem to be an effective way to derive the analytic approximation formulas for credit derivatives, though further development is needed to deal with the pricing problems of credit derivatives. Recently, exotic credit derivatives such as the defautable basket swap valuation problem was studied by Muroi (2003). It also seems important to establish statistical methods to measure credit risk. The computational methods of credit derivatives are progressing rapidly, however, with the statistical methods for empirical investigations of credit risk continuing to be to be challenging for future research.

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